WEDNESDAY'S LECTURE

CHAPTER 4

BRAUER CHARACTERS

CHAPTER 5

BLOCK THEORY

3 Main approaches to representation theory of finite groups

1. Representations

$\rho: G \to GL(V)$

and :

- their characters
  $\chi_\rho : G \longrightarrow K$
  $g \mapsto Tr(\rho(g))$

  if $char(K) = 0$

- their Brauer characters
  $\varphi_\rho : G_{p'} = \{g \in G \mid p \nmid o(g)\} \longrightarrow \mathbb{C}$

  if $char(K) > 0$

2. kG - modules

→ study of the indecomposable / simple / projective / ... modules

→ properties of the module category

3. p-Block theory / algebra approach

→ $kG = B_0 \oplus B_1 \oplus \cdots \oplus B_n$

→ study of the k-algebra structure of kG, resp. of the B_i s

! FIND GOOD ⟷ CONNECTIONS

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Class of permutation kG-modules

← "Lift" kG-modules to char. zero
⇝ associate characters

→ Parametrize many equivalences of block algebras

↓

Computer algebra !

Chapter 4

Brauer Characters

§15. p-MODULAR SYSTEMS

DEF^N: Let $p \in \mathbb{P}$.

(a) A **p-modular system** is a triple of rings $(F, \mathcal{O}, k)$ s.t.:

(1) $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero

(2) $F = \mathrm{Frac}(\mathcal{O})$ $\quad$ ($\mathrm{char}(F) = 0$, too)

(3) $k = \mathcal{O}/J(\mathcal{O})$ is s.t. $\mathrm{char}(k) = p$ $\quad$ (the **residue field** of $\mathcal{O}$)

(b) If both F and k are splitting fields for G, then $(F, \mathcal{O}, k)$ is called a **splitting p-modular system for G**.

Assumption ⊛: From now on we assume that a p-modular system $(F, \mathcal{O}, k)$ is given and is s.t. F contains an $\exp(G)$-th root of unity.
Set $\mathfrak{p} := J(\mathcal{O})$.

$\overset{\text{Brauer}}{\Longrightarrow}$ $(F, \mathcal{O}, k)$ is splitting for G & all its subgroups

EXAMPLE:

- $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$ is a p-modular system
- ⊛ is satisfied if we adjoin an $\exp(G)$-th root of 1 to $\mathbb{Z}_p$

The situation is

$$\begin{array}{ccccc}
\mathbb{Q}_p & \xleftarrow{\text{can}} & \mathbb{Z}_p & \xrightarrow{\text{can}} & \mathbb{F}_p \\
\downarrow{\scriptstyle\text{can}} & & \downarrow{\scriptstyle\text{can}} & & \downarrow{\scriptstyle\text{can}} \\
F & \xleftarrow{\text{can}} & \mathcal{O} & \xrightarrow{\text{can}} & k
\end{array}$$

$\rightsquigarrow$ extend to ring homomorphisms

$$FG \xleftarrow{\text{can}} \mathcal{O}G \xrightarrow{\text{can}} kG$$

and get functors

$$\begin{array}{ccccc}
FG\text{-mod} & \xleftarrow{F\otimes_{\mathcal{O}} -} & \mathcal{O}G\text{-lat} & \xrightarrow{k\otimes_{\mathcal{O}} -} & kG\text{-mod} \\
F\otimes_{\mathcal{O}} L & \longleftarrow & L & \longmapsto & k\otimes_{\mathcal{O}} L
\end{array}$$

'reduction modulo p'

(always possible!)

$$\begin{array}{ccccc} FG\text{-mod} & \xleftarrow{F\otimes_{\mathcal{O}}-} & \mathcal{O}G\text{-lat} & \xrightarrow{k\otimes_{\mathcal{O}}-} & kG\text{-mod} \\ F\otimes_{\mathcal{O}} L & \longleftarrow\!\shortmid & L & \longmapsto & k\otimes_{\mathcal{O}} L \end{array}$$

'reduction modulo $\mathfrak{p}$'

(always possible!)

The other way around:

Def^N: A kG-module M is called liftable (to $\mathcal{O}$, to characteristic 0) if $\exists$ an $\mathcal{O}G$-lattice $\hat{M}$ s.t. $k\otimes_{\mathcal{O}}\hat{M}\cong M$.

⚠ This is rare! But:

THM: (a) Projective kG-modules are liftable.

(b) p-permutation kG-modules are liftable.

(c) [L.-Thévenaz, '17] kG-modules whose k-endomorphism ring is a p-permutation module are liftable.

Consequence: If M is a p-permutation kG-module, then it affords an F-character:

$$M \rightsquigarrow \underbrace{\hat{M}}_{\substack{\text{unique lift}\\ \text{which is}\\ \text{again}\\ p\text{-permutation}}} \rightsquigarrow \chi_M := F\text{-character of } F \otimes_{\mathcal{O}} \hat{M}$$

REM: These ordinary characters χ_M contain a lot of information about the p-permutation kG-modules!

§16. Brauer characters

<u>Recall:</u> k-characters are not good! <u>E.g.:</u> $W = k \oplus \cdots \oplus k$ ($p+1$ times)
$\Rightarrow$ k-character of W = trivial k-character

$\longrightarrow$ need to replace them with different functions in order to obtain a "good character theory" for kG-modules: the 'Brauer characters'.

<u>Note:</u> $\mathcal{O} \overset{can}{\twoheadrightarrow} \mathcal{O}/\mathfrak{p} = k$ induces a bijection $\left\{ \begin{matrix} a\text{-th roots} \\ \text{of } 1 \text{ in } F \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} a\text{-th roots} \\ \text{of } 1 \text{ in } k \end{matrix} \right\}$

$$\hat{\xi} \longrightarrow \xi$$

where $a := \text{l.c.m.}(\, o(g) \mid g \in G_{p'})$.

<u>Diagonalisation Lemma:</u> If $M \in \text{mod}(kG)$ and $\rho_M : G \to GL(M)$ is the assoc. k-repres., then $\forall g \in G_{p'}$ $\exists$ a k-basis B of M s.t.

$$[\rho(g)]_B = \begin{bmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_n \end{bmatrix}$$ with $n := \dim_k M$ and the ξ_i's are $o(g)$-th roots of 1.

DEF^{N}: • Let $M \in \operatorname{mod}(kG)$. Set $n := \dim_k M$.
The **Brauer character afforded** by M is the F-valued function

$$\varphi_M : G_{p'} \longrightarrow F = \operatorname{Frac}(\mathcal{O})$$
$$g \longmapsto \hat{\xi}_1 + \cdots + \hat{\xi}_n$$

where the ξ_i are as in the DL.

• φ_M is **irreducible** if M is simple.

• $\operatorname{IBr}_p(G) =: \{\text{irred. Brauer characters of } G\}$.

Back to reduction modulo $\mathfrak{p}$:

LEM: Let $V \in \mathrm{mod}(FG)$ with F-character $\chi_V : G \to F,\ g \mapsto \mathrm{Tr}(\rho_V(g))$.

Then:
(1) $\exists$ an $\mathcal{O}G$-lattice L s.t. $V \cong F \otimes_{\mathcal{O}} L$ ("L is an $\mathcal{O}$-form of V")

(2) $\chi_V|_{G_{p'}} = \varphi_{k \otimes_{\mathcal{O}} L}$ ("the reduction modulo $\mathfrak{p}$ of χ_V")

(3) If $V \in \mathrm{Irr}(FG)$, then $\exists$ integers $d_{\chi_V \varphi} \geq 0$ s.t.

$$\chi_V|_{G_{p'}} = \sum_{\varphi \in \mathrm{IBr}_p(G)} d_{\chi_V \varphi}\, \varphi$$

* $$\mathrm{Dec}_p(G) := \left(d_{\chi\varphi} \right)_{\substack{\chi \in \mathrm{Irr}_F(G) \\ \varphi \in \mathrm{Irr}_k(G)}}$$

is the *p-decomposition matrix of G*

* $$C := C_p(G) := \mathrm{Dec}_p(G)^{tr} \cdot \mathrm{Dec}_p(G)$$

is the *Cartan matrix of G*

Chapter 5

Block Theory

Assume: $\Delta \in \{F, \mathcal{O}, k\}$; G, H are finite groups.

OBSERVE: If M is a $(\Delta G, \Delta H)$-bimodule, then M can be seen as a left $\Delta[G \times H]$-module via

$$(g,h) \cdot m := g \cdot m \cdot h^{-1} \qquad \forall g \in G, \forall h \in H, \forall m \in M.$$

§ 17. p-Blocks

Blocks of ΛG:

* ΛG has a unique decomposition $\quad \Lambda G = B_0 \oplus \cdots \oplus B_n$ into indecomposable $(\Lambda G, \Lambda G)$-subbimodules. These are called the **blocks of ΛG**.

* Decomposing
$$1_{\Lambda G} = \underset{\substack{\uparrow \\ B_0}}{e_0} + \cdots + \underset{\substack{\uparrow \\ B_n}}{e_n}$$
we have $e_i = 1_{B_i}$ and $B_i = \Lambda G e_i$ $\forall\, 0 \leq i \leq n$, where each e_i is a primitive idempotent in $Z(\Lambda G)$ and $e_i e_j = \delta_{ij}$ $\forall\, 0 \leq i,j \leq n$.

Belonging to a block:

Each indecomposable ΛG-module can be assigned to a block:
$$M = 1_{\Lambda G} \cdot M = e_0 \cdot M \oplus \cdots \oplus e_n \cdot M$$
$$\Longrightarrow \quad \exists\, 0 \leq i \leq n \text{ s.t. } \begin{cases} e_i M = M \\ e_j M = 0 & \text{if } j \neq i \end{cases}$$
$\rightsquigarrow$ We say that **M belongs to the block B_i**.

THE PRINCIPAL BLOCK: is the block of ΛG containing the the trivial module Λ. **NOTA:** $B_0(\Lambda G)$.

BLOCKS OF FG?

FG is semisimple $\Longrightarrow$ the block decomposition of FG is given by the Artin-Weddenburn thm.

So, the blocks are matrix algebras and can be labelled by $\mathrm{Irr}(FG)$.

BLOCKS OF $\mathcal{O}G$ and kG?

The lifting-of-idempotents thm tells us that $\mathcal{O}G \twoheadrightarrow kG$ induces a bijection

$$\left\{\begin{array}{c}\text{primitive}\\ \text{idempotents}\\ \text{of } Z(\mathcal{O}G)\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{primitive}\\ \text{idempotents}\\ \text{of } Z(kG)\end{array}\right\}$$

$$e \longmapsto \bar{e}$$

$\Rightarrow$ $\exists$ a bijection between the blocks of $\mathcal{O}G$ and the blocks of kG

$$\begin{array}{ccc} \mathcal{O}G & = & B_0 \oplus \cdots \oplus B_n \\ \Big\downarrow{\scriptstyle \text{can}} & & \\ kG & = & \bar{B}_0 \oplus \cdots \oplus \bar{B}_n \end{array} \qquad (\text{with } \bar{B}_i = kG\bar{e}_i)$$

<u>Now:</u> A **p-block of G** is the specification of a block of $\mathcal{O}G$, or of the corresponding block of kG.

<u>NOTA:</u> $Bl_p(kG) = \{\text{p-block of } kG\}$ / $Bl_p(\mathcal{O}G) = \{\text{p-blocks of } \mathcal{O}G\}$

DEFECT GROUPS:

A defect group of a p-block $B \in Bl_p(\mathcal{O}G)$ is a vertex of B seen as a left $\mathcal{O}[G\times G]$-module.

(Or equiv. a vertex of $\overline{B}$ as a left $k[G\times G]$-module.)

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PROPERTIES:

(1) Defect groups form a conjugacy class of p-subgroups of G

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(2) If D is a defect group of $B \in Bl_p(kG)$, then any indec. kG-module belonging to B is D-projective, hence has a vertex contained in D.

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(3) (2) $\Rightarrow$ the defect groups of $B_0(kG)$ are precisely $Syl_p(G)$.

Def^N: Let $H \leq G$ and let $b \in Bl_p(kH)$.

A p-block $B \in Bl_p(kG)$ corresponds to b

:<=> $b \mid B\downarrow^{G\times G}_{H\times H}$ and b is the unique block of kG with this property.

Nota: $B = b^G$

If such a block exists, we say that b^G is defined.

THM: [Brauer's correspondence]

Let $D \leq G$ be a p-subgroup and let $H \leq G$ s.t. $H \geq N_G(D)$.
Then $\exists$ a bijection

$$\left\{\begin{array}{l}\text{blocks of } kH \text{ with} \\ \text{defect group } D\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{blocks of } kH \text{ with} \\ \text{defect group } D\end{array}\right\}$$

$$b \longmapsto b^G$$

Proof: This is just a particular case of the Green correspondence!

#

§18. Equivalences of block algebras

Basic Question: (Open)

> Which k-algebras occur as p-blocs of finite groups?

Conjectural Answer: If a defect group P is fixed only finitely many ... up to a good notion of equivalence. More accurately:

Donovan's (Puig's) Conjecture: ['70's / '80's] Let P be a p-group.

> There are only finitely many possible splendid Morita equivalence classes for p-blocks of finite groups with defect group isomorphic to P.

'Donovan' holds for: a (fairly long) list of "small" P's.

> Wiki site by Charles Eaton:

https://wiki.manchester.ac.uk/blocks/index.php

'Puig' holds for: $P \cong C_{p^n}$ (cyclic), $P \cong C_2 \times C_2$ ($p=2$)

Let $B \in Bl_p(RG)$, let $C \in Bl_p(RH)$ with $R \in \{\mathcal{O}, k)$

DEF^N: B and C are Morita equivalent (written $B \sim_M C$) iff $mod(B)$ and $mod(C)$ are equivalent as (R-linear) categories.

MORITA'S THM: TFAE:

(1) $B \sim_M C$

(2) $\exists$ a (B,C)-bimodule M and a (C,B)-bimodule N s.t.

$M \otimes_C N \cong B$ as (B,B)-bimodule

$N \otimes_B M \cong C$ as (C,C)-bimodule.

(NOTE: $N = M^\vee$)

DEF^N: Furthermore, if $B \sim_M C$ via a (B,C)-bimodule M, then the Morita equivalence is called:

* a splendid Morita equivalence (or a source-algebra equivalence) iff M viewed as a $k[G\times H]$-module is a p-permutation module. NOTA: $B \sim_{SM} C$

* an endo-permutation source Morita equivalence (or a basic Morita equivalence) iff M seen has $k[G\times H]$-module has a source T s.t. $\mathrm{End}_k(T)$ is a permutation module.

NOTA: $B \sim_{EPS} C$

NOTE:

Defect groups are preserved by splendid Morita equivalences and by endo-permutation source Morita equivalence.

NOT by Morita equivalence !!

There are many variations / other types of equivalences relevant to block theory:

E.g.:
* stable equiv. of Morita type: equiv. of the stable module categories
* Rickard equivalences: equiv. of the derived categories
* splendid Rickard equivalences / p-permutation equivalences (given by tensoring with complexes of p-permutatio $k[G\times H]$-modules.)

Examples:

① • Isomorphic blocks as k-algebras are Morita equivalent.
• Blocks with a common defect group D, isomorphic as interior D-algebras are splendidly Morita equivalent.

①' In particular: Inflation from $G/O_{p'}(G)$ to G yields

$$B_0(kG) \sim_{SM} B_0(k[G/O_{p'}(G)])$$

(as $O_{p'}(G)$ always acts trivially on the principal block)

② "<u>Fong-Reynolds</u>": Let $H \trianglelefteq G$, $b \in Bl_p(kH)$, $T := \mathrm{Stab}_G(b)$, then $\exists$ bijection

$$\begin{aligned} Bl_p(T \mid b) &\xrightarrow{\sim} Bl_p(G \mid b) \\ B &\longmapsto B^G \end{aligned}$$

and $M := 1_{B^G} \cdot kG \cdot 1_B$ realises a splendid Morita equivalence between B and B^G.

3 'Fong's 2nd reduction' is an endo-permutation source Morita equivalence.

SPLENDID MORITA EQUIVALENCE

$\Downarrow$

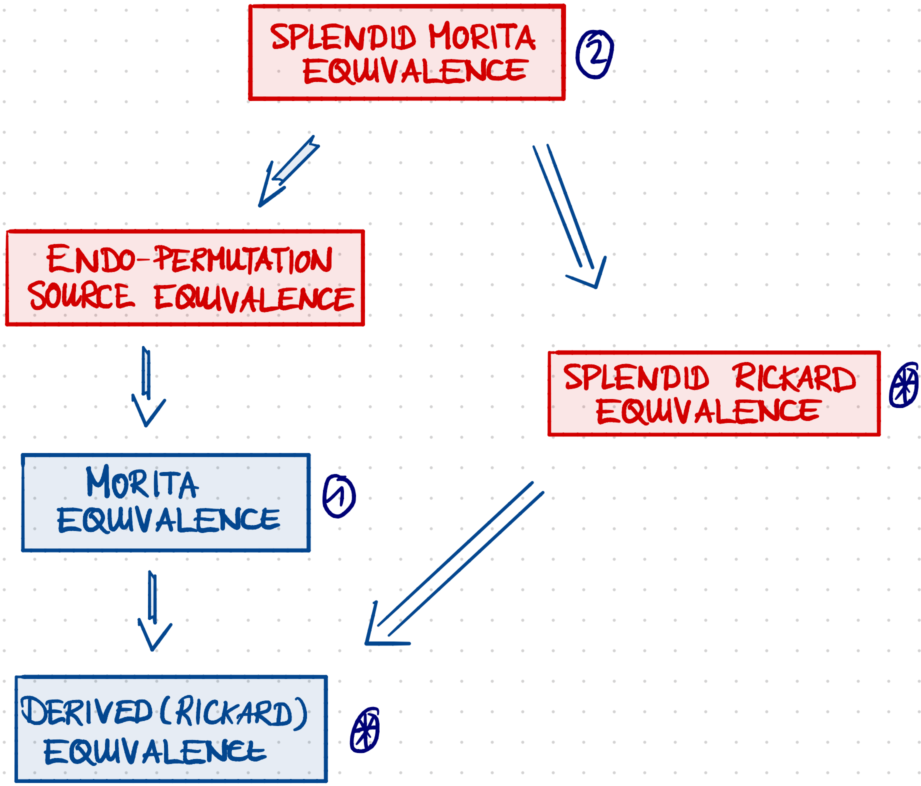
ENDO-PERMUTATION SOURCE EQUIVALENCE

$\Downarrow$

MORITA EQUIVALENCE

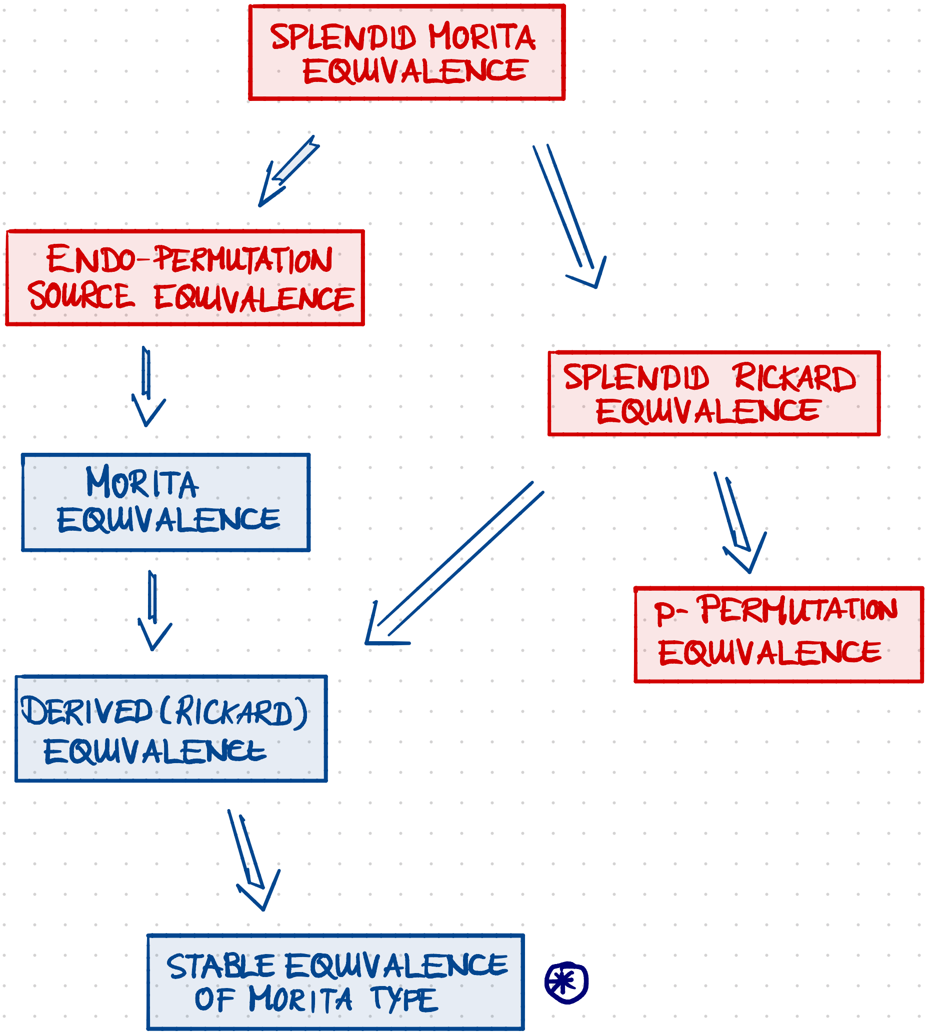
$\Downarrow$

IN RED: involve (p-) permutation modules



IN RED: involve (p-) permutation modules

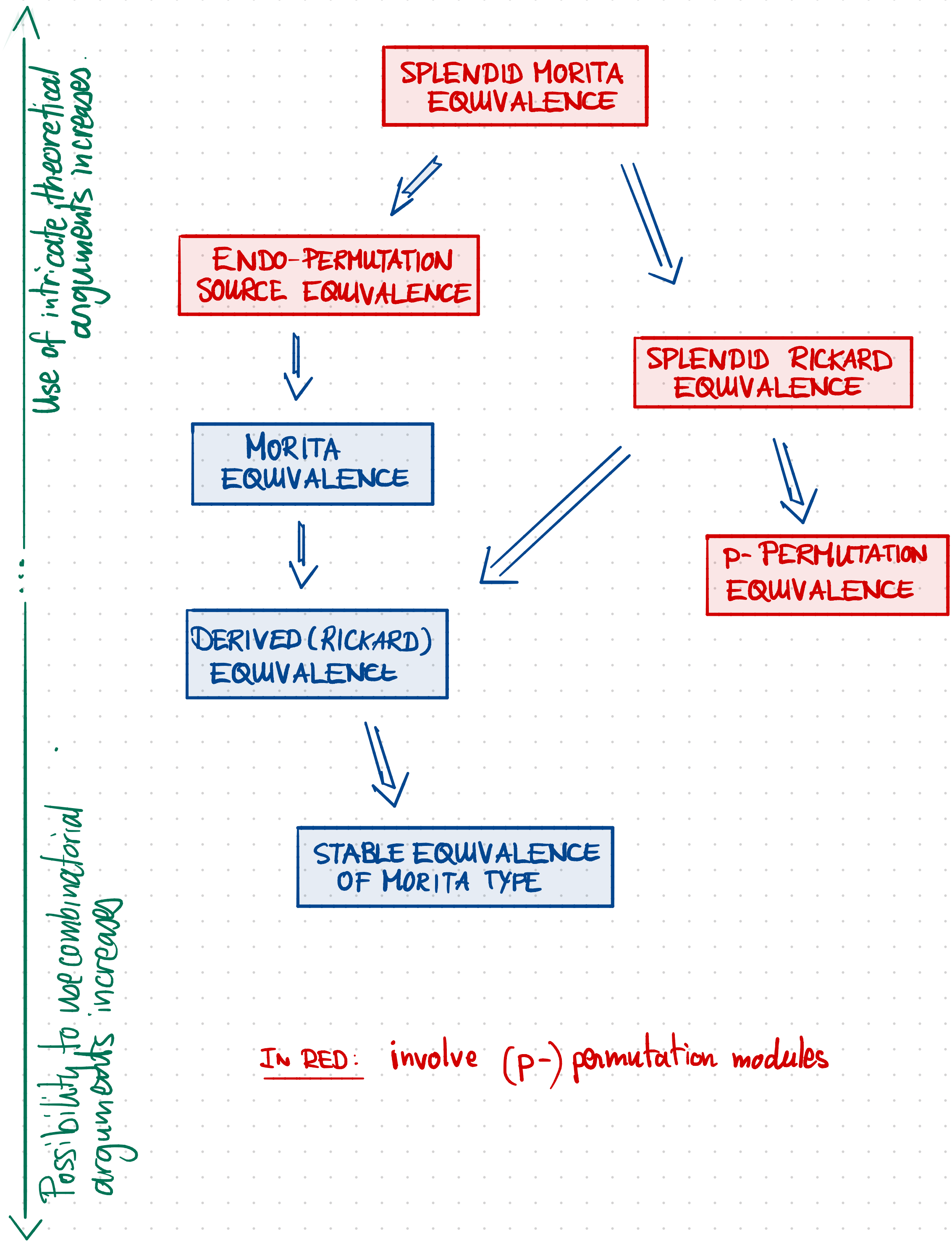
⊛ Derived category versions of ① and ②: replace modules by complexes.

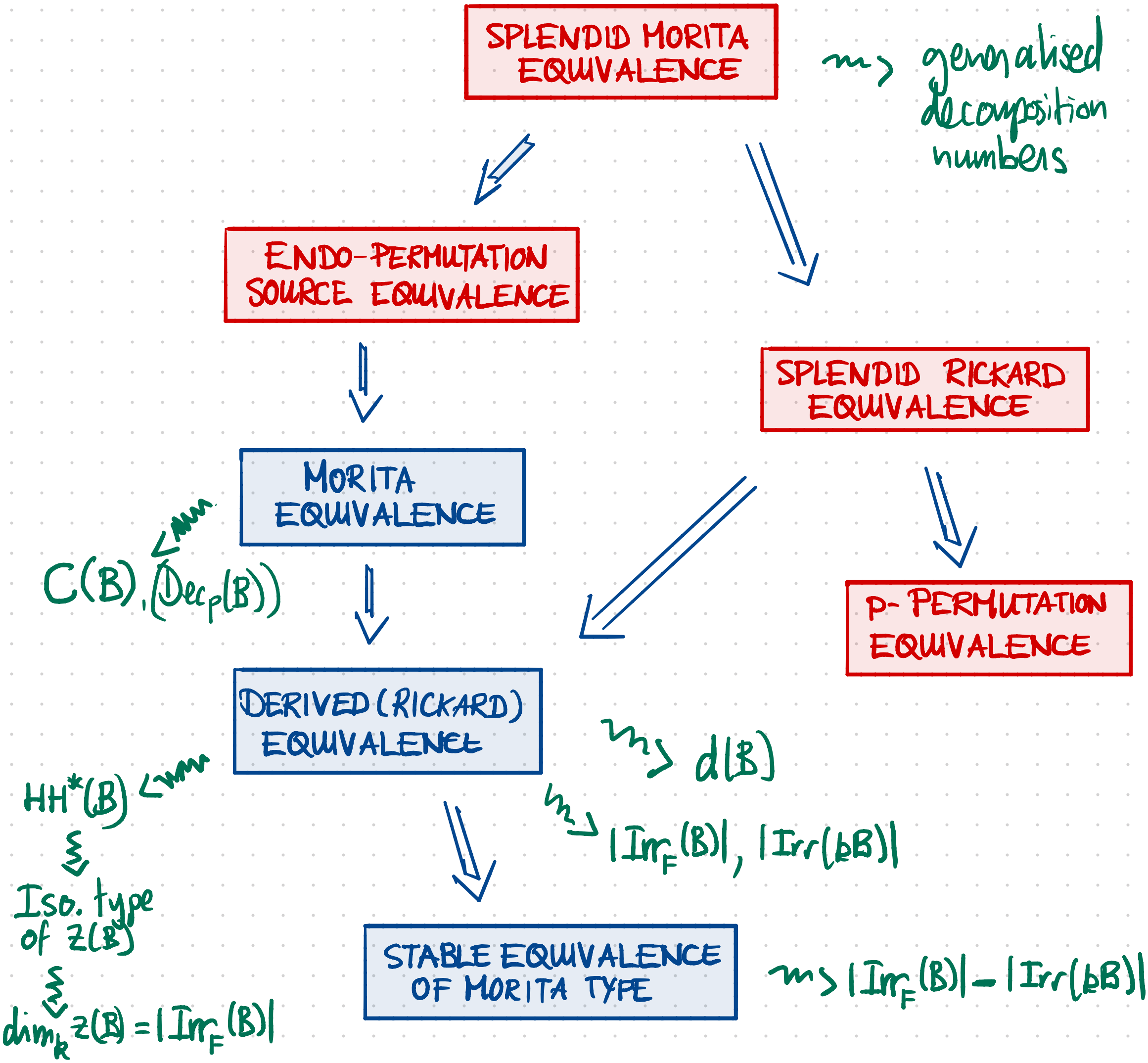


<u>IN RED:</u> involve (p-) permutation modules

⊛ Modify DEF^N of a Morita Equiv.:

$$M \otimes_B N \cong A \oplus (\text{proj}) \qquad N \otimes_A M \cong B \oplus (\text{proj})$$


SPLENDID MORITA EQUIVALENCE
ENDO-PERMUTATION SOURCE EQUIVALENCE
SPLENDID RICKARD EQUIVALENCE
MORITA EQUIVALENCE
p-PERMUTATION EQUIVALENCE
DERIVED (RICKARD) EQUIVALENCE
STABLE EQUIVALENCE OF MORITA TYPE
Use of intricate theoretical arguments increases.
Possibility to use combinatorial arguments increases.
IN RED: involve (p-) permutation modules



IN RED: involve (p-) permutation modules

Many open problems in modular representation theory are concerned with the influence of the structure of the defect groups on the structure of the block. E.g.

Brauer's $k(B)$-conjecture

Let $B \in \mathrm{Bl}_p(kG)$ with defect group D. Then $\#\mathrm{Irr}(B) \leq |D|$.

Broué's abelian defect group conjecture

Let $B \in \mathrm{Bl}_p(kG)$ be a block with an abelian defect group. Then B and its Brauer correspondent in $N_G(D)$ are derived (Rickard) equivalent.