CONVENTIONS!

Unless otherwise stated:

* groups are finite
* rings are assoc. with a 1
* modules are finitely generated left modules

* G is a group
* R is an arbitrary assoc. ring with a 1, and
  R° is the regular module on R
* K is a field with char(K) <u>arbitrary</u>
  k —— " —— with char(k) $=: p > 0$ $\rightsquigarrow p \in \mathbb{P}$
  and char(k) $\mid |G|$
* A is a finite-dimensional K-algebra

Monday's lecture

Chapter 1

Representations of finite groups

§1. Linear Representations

Def^N: A *K-representation of G* is a group homomorphism
$$\rho : G \longrightarrow GL(V) := \mathrm{Aut}_K(V)$$
where $V \cong K^n$ $(n \in \mathbb{Z}_{\geq 0})$ is a K-vector space.

Moreover: * n is the *degree of ρ*;

* ρ is $\left|\begin{array}{l}\text{an ordinary} \\ \text{a modular}\end{array}\right|$ representation if $\left|\begin{array}{l}\mathrm{char}(K) \nmid |G| \\ \mathrm{char}(K) \mid |G|\end{array}\right.$.

Note: choosing a K-basis of V yields:

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, dashed, "\circlearrowleft"] & GL(V) \arrow[d, phantom, "\cong" rotate=-90] & \varphi \arrow[d, mapsto] \\ & GL_n(K) & (\varphi)_B \end{tikzcd}$$

Hence: any K-representation defines a "*matrix representation*" and conversely.

EXAMPLES:

(a) $$\rho : \begin{array}{ccc} G & \longrightarrow & GL(k) \cong k^\times \\ g & \longmapsto & \mathrm{Id}_k \end{array}$$ is a k-representation of G

— the trivial representation of G

(b) $\begin{cases} X \text{ finite } G\text{-set} & (\text{with } \cdot : G\times X \to X, g \mapsto g\cdot x \text{ left action}) \\ V \text{ } k\text{-vector space} & \text{with } k\text{-basis } \{e_x \mid x \in X\} \end{cases}$

$$\Longrightarrow \quad \rho_X : \begin{array}{ccc} G & \longrightarrow & GL(V) \\ g & \longmapsto & \rho_X(g) : V \longrightarrow V,\ e_x \mapsto e_{g\cdot x} \end{array}$$

is a k-representation of G — the permutation representation of G on X

Two particular cases:

① $G = S_n$, $X = \{1, 2, \dots, n\}$, $\cdot$ = natural action

$\rightsquigarrow$ ρ_X = natural representation of S_n

② $X = G$, $\cdot$ = mult. in G, then ρ_X is called the regular representation of G. We write $\rho_X =: \rho_{reg}$

Morphisms

Let $\rho_i : G \to GL(V_i)$ $(i \in \{1,2\})$ be k-repres.

* A **morphism** between ρ_1 and ρ_2 is a k-homomorphism $\alpha : V_1 \to V_2$ s.t.

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & V_1 \arrow[d, "\alpha"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

commutes $\forall g \in G$.

* If α is a k-iso., then ρ_1 and ρ_2 are said to be **equivalent** and we write $\rho_1 \sim \rho_2$.

$\sim$ is an equivalence relation

$\longrightarrow$ enough to study k-representations up to equivalence !

Substructures

* A K-subspace $W \subseteq V$ is **G-invariant** iff

$$\rho(g)(W) \subseteq W \qquad \forall\, g \in G.$$

* ρ is **irreducible** if it has <u>exactly</u> two G-invariant subspaces, i.e. 0 and V.

* $0 \subsetneq W \subsetneq V$ G-invariant K-subspace

$$\Rightarrow \quad \rho_W : \begin{aligned} G &\longrightarrow GL(W) \\ g &\longmapsto \rho(g)|_W \end{aligned}$$

K-subrepresentation of ρ assoc. to W

§2. The Group Algebra and Its Modules

From now on, we want to see K-representations as modules.

Def^N: The group algebra of G over K is the K-algebra

$$KG := \{ \sum_{g\in G} \lambda_g g \mid \lambda_g \in K \ \forall g \in G \}$$

with addition

$$\sum_{g\in G} \lambda_g g + \sum_{g\in G} \lambda'_g g = \sum_{g\in G} (\lambda_g + \lambda'_g) g$$

and multiplication

$$\left(\sum_{g\in G} \lambda_g g\right) \cdot \left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} (\lambda_g \mu_h) gh .$$

(i.e. mult. in G is extended by K-bilin. to KG)

Notice:

- $1_{KG} = 1_G$
- $\dim_K(KG) = |G|$
- K field ⟹ KG left Artinian ⟹ Hopkins' thm holds: $M \in \mathrm{mod}(KG)$ is f.g. ⟺ M has a composition series
- KG is commutative ⟺ G is abelian
- sidedness of modules not an issue: $m \cdot g := g^{-1} \cdot m$

PROP

* $\rho : G \to GL(V)$ K-representation $\Longrightarrow$ V is a KG-module via
$$\cdot : G \times V \longrightarrow V$$
$$(g, v) \longmapsto g \cdot v := \rho(g)(v)$$

* $(V, +, \cdot)$ KG-module $\Longrightarrow$
$$\rho_V : G \longrightarrow GL(V)$$
$$g \longmapsto \rho_V(g) : V \to V$$
$$v \mapsto \rho_V(g) := g \cdot v$$

EXAMPLE

Via the Prop. the trivial representation of G corresponds to the "trivial KG-module", i.e. K itself seen as a KG-module via the G-action
$$\cdot : G \times K \longrightarrow K$$
$$(g, \lambda) \longmapsto g \cdot \lambda = \lambda$$
extended by K-linearity to KG.

EXAMPLE

$$\rho_{reg} \longleftrightarrow \text{regular module } (KG)^\circ$$

(In the sequel, we drop the "$\circ$" symbole!)

K-Representations		KG-Modules
K-representation of G	$\longleftrightarrow$	KG-module
degree	$\longleftrightarrow$	K-dimension
homomorphism of K-representations	$\longleftrightarrow$	homomorphism of KG-modules
equivalent K-representations	$\longleftrightarrow$	isomorphism of KG-modules
subrepresentation	$\longleftrightarrow$	KG-submodule
direct sum of representations $\rho_{V_1} \oplus \rho_{V_2}$	$\longleftrightarrow$	direct sum of KG-modules $V_1 \oplus V_2$
irreducible representation	$\longleftrightarrow$	simple (= irreducible) KG-module
the trivial representation	$\longleftrightarrow$	the trivial KG-module K
the regular representation of G	$\longleftrightarrow$	the regular KG-module KG°
completely reducible K-representation	$\longleftrightarrow$	semisimple KG-module (= completely reducible)
every K-representation of G is completely reducible	$\longleftrightarrow$	KG is semisimple
...		...

§3. OPERATIONS ON GROUPS AND MODULES

Q.? How can we construct new KG-modules from old ones?

Tensors, Hom's, duals Let M, N be KG-modules.

(a) $M \otimes_K N$ becomes a KG-module via the *diagonal action of G*, i.e.

$$\begin{aligned} \cdot : G \times (M \otimes_K N) &\longrightarrow M \otimes_K N \\ (g, m \otimes n) &\longmapsto g \cdot (m \otimes n) := gm \otimes gn \end{aligned}$$

(b) $\mathrm{Hom}_K(M,N)$ becomes a KG-module via the *conjugation action of G*, i.e.

$$\begin{aligned} \cdot : G \times \mathrm{Hom}_K(M,N) &\longrightarrow \mathrm{Hom}_K(M,N) \\ (g, f) &\longmapsto g \cdot f : M \longrightarrow N \\ & \qquad\quad m \longmapsto (g \cdot f)(m) := g \cdot f(g^{-1} \cdot m) \end{aligned}$$

(c) $M^* := \mathrm{Hom}_K(M, K)$ is a KG-module via (b).
(i.e. with $N = K$, the trivial KG-module)

One can also let the group vary! Using 'changes of the base ring', we obtain:

① RESTRICTION

$H \leq G \Rightarrow H \hookrightarrow G, h \mapsto h$ extends K-linearly to a K-algebra homomorphism

$$i : KH \hookrightarrow KG$$

If $M \in \mathrm{mod}(KG)$, then M becomes a KH-module via a change of the base ring along i, which we denote by $\mathrm{Res}^G_H(M)$ or $M\downarrow^G_H$.

② INFLATION

$U \trianglelefteq G \Rightarrow G \twoheadrightarrow G/U, g \mapsto gU$ extends K-linearly to a K-algebra homomorphism

$$\pi : KG \twoheadrightarrow K[G/U]$$

Hence: any $K[G/U]$-module becomes a KG-module via a change of the base ring along π, which we denote by $\mathrm{Inf}^G_{G/U}(M)$.

PROPERTIES:
- Res and Inf commute with $\oplus$ and $(-)^*$
- Res is transitive

...

A 3rd operation is given by extending the scalars from a subgroup:

③ INDUCTION

Let $H \leq G$ and $M \in \mathrm{mod}(KH)$. Seeing KG as a (KG, KH)-bimodule, we may define $\mathrm{Ind}_H^G(M) := KG \otimes_{KH} M$.

$$\mathrm{Ind}_H^G(M) = M\uparrow_H^G$$

EXAMPLE

(a) $H = \{1\}$, $M = K \implies K\uparrow_{\{1\}}^G = KG \otimes_K K \cong KG$

(b) assoc. of $\otimes \implies (M\uparrow_L^H)\uparrow_H^G \cong M\uparrow_L^G \quad \forall\ L \leq H \leq G$

REM: $KG \otimes_{KH} M = \bigoplus_{g \in [G/H]} g \otimes M$ as K-vect. space

$$\implies \boxed{\dim_K(M\uparrow_H^G) = |G:H| \cdot \dim_K(M)}$$

IMPORTANT RELATIONS between the above operations are:

* **FROBENIUS RECIPROCITY** (= "biadjunction of Res and Ind")

$$\mathrm{Hom}_{kH}(M, N\downarrow^G_H) \cong \mathrm{Hom}_{kG}(M\uparrow^G_H, N)$$

$\uparrow$ k-iso $\downarrow$

$$\mathrm{Hom}_{kH}(N\downarrow^G_H, M) \cong \mathrm{Hom}_{kG}(N, M\uparrow^G_H)$$

* $$(M \otimes_k N)\uparrow^G_H \cong M\uparrow^G_H \otimes_k N$$

$\uparrow$ kG-iso. $\downarrow$

$$\mathrm{Hom}_k(M, N\downarrow^G_H)\uparrow^G_H \cong \mathrm{Hom}_k(M\uparrow^G_H, N)$$

* **THE MACKEY FORMULA** Let $H, L \leq G$, let $M \in \mathrm{mod}(kL)$.

Then, as kH-modules,

$$M\uparrow^G_L\downarrow^G_H \cong \bigoplus_{g \in [H\backslash G/L]} \left({}^gM\downarrow^{{}^gL}_{H\cap {}^gL}\right)\uparrow^H_{H\cap {}^gL}.$$

(where ${}^gM$ is $g\otimes M$ seen as a left $k({}^gL)$-module with $(glg^{-1})\cdot(g\otimes m) = g\otimes lm$)

BREAK: ⚠ In general $\mathrm{mod}(KG)$ is wild

$\longrightarrow$ not possible to classify KG-modules !!

$\longrightarrow$ Need to restrict our attention to "reasonable" classes of KG-modules!

In the sequel, we take a closer look at:

simple semi-simple projective permutation/ p-permutation	KG-modules

Chapter 2

Simplicity and Semisimplicity

NOTA: $\mathrm{Irr}(R) := \{\text{simple } R\text{-modules}\}/_{\cong}$

§4. Schur's Lemma

LEM (Schur)

(a) $V, W \in \mathrm{Irr}(R) \implies \begin{cases} (1) & \mathrm{End}_R(V) \text{ is a skew-field} \\ (2) & V \not\cong W \implies \mathrm{Hom}_R(V,W) = 0 \end{cases}$

(b) If $k = \bar{k}$, A finite dim'l k-algebra and $V \in \mathrm{Irr}(A)$ s.t. $\dim_k V < \infty$, then

$$\mathrm{End}_A(V) = \{\lambda \mathrm{Id}_V \mid \lambda \in k\} \cong K.$$

Proof:

(a) trivial!

(b) "$\supseteq$" ✓ ; "$\subseteq$" given $f \in \mathrm{End}_A(V)$, have $f = \lambda \cdot \mathrm{Id}_V$ for λ eigenvalue of f.

REM: In (b), the Ⓗ that $K=\overline{K}$ is in gen. too strong!
We often replace this Ⓗ with the Ⓗ that

$\boxed{A \text{ is split}}$ $:\Longleftrightarrow$ $\mathrm{End}_A(V) \cong K$ $\forall$ f.d. simple A-module V.

This leads to the following definition:

DEF^N: K is a splitting field for G iff KG is a split K-algebra.

From now on, we assume that

$\boxed{\begin{array}{l} K \text{ is a splitting field for } G \\ \text{(and all other groups involved)}. \end{array}}$

§5. Artin-Wedderburn

Def^N: Let $S \in \mathrm{Irr}(R)$. If M is a semisimple R-module, then the S-homogeneous part of M, written $S(M)$, is the sum of all simple R-submodules of M isomorphic to S.

THM: [Wedderburn]

If R is a semisimple ring, then:

(1) $S \in \mathrm{Irr}(R) \Rightarrow S(R^\circ) \neq 0$ and so $|\mathrm{Irr}(R)| < \infty$;

(2) $R^\circ = \bigoplus_{S \in \mathrm{Irr}(R)} S(R^\circ)$, where each $S(R^\circ)$ is a simple left Artinian ring

Note: R semisimple $\Rightarrow$ R° has a composition series

$$\xRightarrow{\text{Jordan-Hölder}} \quad R^\circ = \bigoplus_{S \in \mathrm{Irr}(R)} S(R^\circ) = \bigoplus_{S \in \mathrm{Irr}(R)} \bigoplus_{i=1}^{n_S} S$$

for uniquely determined $n_S \in \mathbb{Z}_{>0}$.

THM: [Artin-Wedderburn]

R is a semisimple ring $\Longrightarrow$ $R \cong \prod_{S \in \mathrm{Irr}(R)} M_n(D_s)$

where $D_s := \mathrm{End}_R(S)^{\mathrm{opp}}$ is a division ring.

Assuming now that $\boxed{R = A \text{ is a split } K\text{-algebra}}$ we obtain as a corollary:

THM: If A is semisimple and $S \in \mathrm{Irr}(A)$, then:

(a) $S(A^\circ) \cong M_{n_s}(K)$ and $\dim_K(S(A^\circ)) = n_s^2$

(b) $\dim_K(S) = n_s$

(c) $\dim_K(A) = \sum_{S \in \mathrm{Irr}(A)} \dim_K(S)^2$

(d) $|\mathrm{Irr}(A)| = \dim_K(Z(A))$

COR 1: $|\mathrm{Irr}(A)| = \dim_K(Z(A/J(A)))$.

Proof: A and $\underbrace{A/J(A)}_{J\text{-s.s.} \Rightarrow \text{s.s.}}$ have the same simple modules

so (d) of the THM yields the claim! #

COR 2: A commutative $\Longrightarrow$ $\dim_K(S) = 1 \quad \forall S \in \mathrm{Irr}(A)$

Proof: $A = Z(A) \Rightarrow |\mathrm{Irr}(A)| \overset{(d)}{=} \dim_K(A) \overset{(c)}{=} \sum_{S \in \mathrm{Irr}(A)} \underbrace{\dim_K(S)^2}_{\geq 1}$ #

Applying these results to $A = KG$, we obtain for example:

$\boxed{\begin{array}{lll} * & |\mathrm{Irr}(KG)| < \infty & (\text{COR } 1) \\ * & G \text{ abelian} \overset{(\text{COR } 2)}{\Longrightarrow} \dim_K(S) = 1 & \forall S \in \mathrm{Irr}(KG) \end{array}}$

and also :

COR 3: Let $p \in \mathbb{P}$. If G is a p-group and $\mathrm{char}(K) = p$, then
$$\mathrm{Irr}(KG) = \{K\}.$$

<u>Proof</u>: "$\supseteq$" Clear: the trivial KG-module is simple.

"$\subseteq$" $$|\mathrm{Irr}(KG)| \underset{\text{(d) of THM}}{=} \dim_K Z(\underbrace{KG/J(KG)}_{\cong K \text{ as } J(KG) = \text{augmentation ideal of } KG \text{ in this case}})$$
$$= \dim_K(K) = 1$$

#

§6. SEMISIMPLICITY OF THE GROUP ALGEBRA

Q.? When is KG semisimple?

THM: [Maschke and its Converse]

$$\mathrm{Char}(K) \nmid |G| \iff KG \text{ is semisimple}$$

Proof: "$\Rightarrow$": standard!

"$\Leftarrow$": can be proved using Artin-Wedderburn!

COR: $\mathrm{Char}(K) \nmid |G| \Rightarrow |G| = \sum_{S \in \mathrm{Irr}(G)} \dim_K(S)^2$

Proof: $\sum_{S \in \mathrm{Irr}(G)} \dim_K(S)^2 = \dim_K(KG) = |G|$ #