A SHORT INTRODUCTION TO THE MODULAR REPRESENTATION THEORY OF FINITE GROUPS

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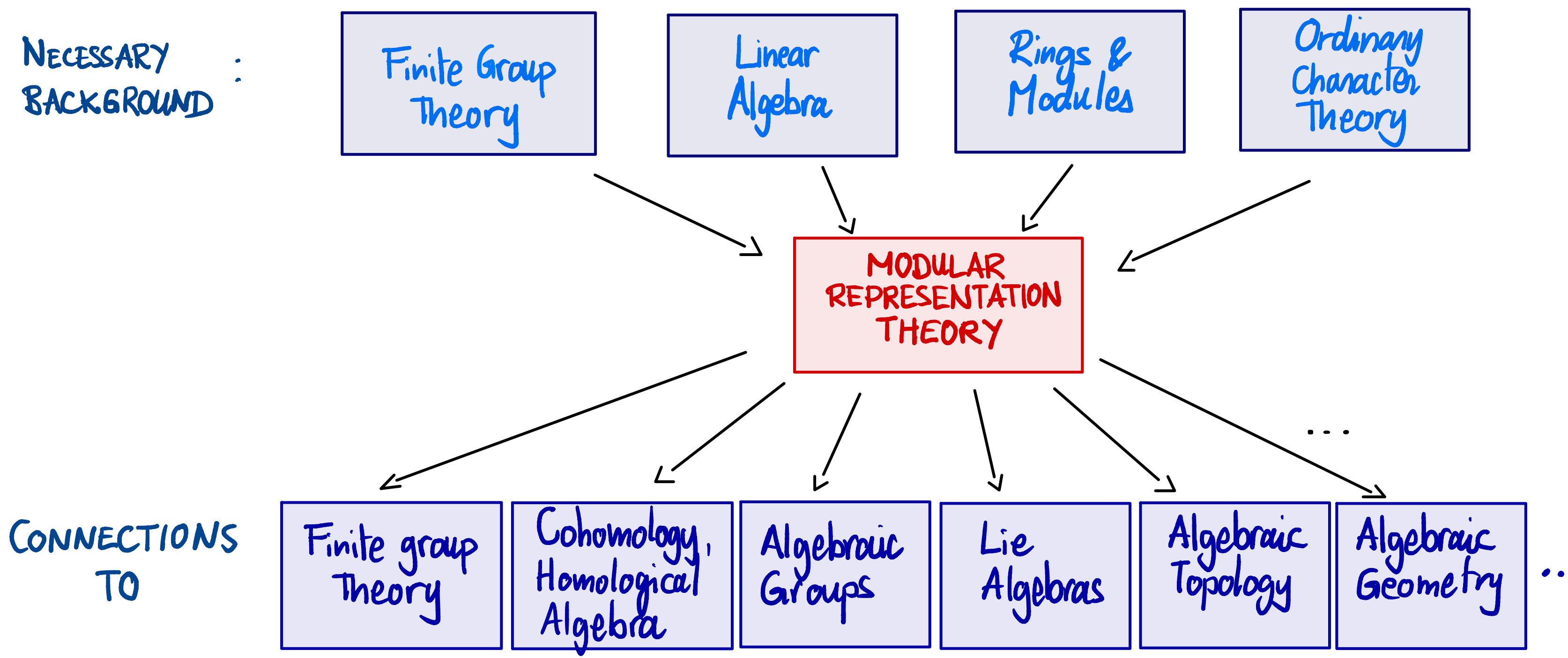
0. Introduction

BEFORE AND AFTER



WARNING :

Modular representation theory of finite groups is a beautiful topic with deep theoretical results / questions / answers (!),

but it is

HARD (VERY HARD) !

→ Often : <u>not</u> much room for combinatorics ! :<

3 Main approaches to representation theory of finite groups

1. Representations

$\rho : G \to GL(V)$

and:

- their characters
  $\chi_\rho : G \longrightarrow K$
  $g \mapsto Tr(\rho(g))$

  if $char(K) = 0$

- their Brauer characters
  $\varphi_\rho : G_{p'} = \{g \in G \mid p \nmid o(g)\} \longrightarrow \mathbb{C}$

  if $char(K) > 0$

2. kG-modules

→ study of the indecomposable / simple / projective / ... modules

→ properties of the module category

3. p-Block theory / algebra approach

→ $kG = B_0 \oplus B_1 \oplus \cdots \oplus B_n$

→ study of the k-algebra structure of kG, resp. of the B_i s

! Find good ⟷ connections

! Find good ⟷ connections

Class of permutation kG-modules

← "Lift" kG-modules to char. zero ⇝ associate characters

→ Parametrize many equivalences of block algebras

↓ Computer algebra!

Ordinary Representation Theory $\longrightarrow$ through characters

Def^N: If $\rho : G \longrightarrow GL(V)$ is a K-representation, then

$$\chi_\rho : \begin{array}{l} G \longrightarrow K \\ g \mapsto Tr(\rho(g)) \end{array}$$

is the K-character of associated to ρ. Moreover χ_ρ is irreducible if ρ is.

Standard Results for $K = \mathbb{C}$:

(1) $\rho_1 \sim \rho_2 \iff \chi_{\rho_1} = \chi_{\rho_2}$

(2) Characters are class functions: $\chi_\rho(hgh^{-1}) = \chi_\rho(g) \quad \forall g, h \in G$.

(3) $Irr_{\mathbb{C}}(G) := \{\text{irreducible } \mathbb{C}\text{-characters of } G\}$ is finite. In fact,

$$|Irr_{\mathbb{C}}(G)| = \#\text{ conjugacy classes of } G$$

(4) $\chi \in Irr_{\mathbb{C}}(G) \implies \chi(1) \mid |G|$

(5) $\sum_{\chi \in Irr_{\mathbb{C}}(G)} \chi(1)^2 = |G|$

(6) G is abelian $\iff \chi(1) = 1 \quad \forall \chi \in Irr_{\mathbb{C}}(G)$

(7) The character table $(\chi(g))_{\substack{\chi \in Irr_{\mathbb{C}}(G) \\ g \in RepConjCl(G)}}$ contains a lot of information about G.