Wednesday. Chapter 4. p-Modular Systems and Brauer Characters

R. Brauer started in the late 1920's a systematic investigation of group representations over fields of positive characteristic. In order to relate group representations over fields of positive characteristic to character theory in characteristic zero, Brauer worked with a triple of rings $(F, \mathcal{O}, k)$, called a *p-modular system*, and consisting of a complete discrete valuation ring $\mathcal{O}$ with a residue field $k := \mathcal{O}/J(\mathcal{O})$ of prime characteristic p and fraction field $F := \mathrm{Frac}(\mathcal{O})$ of characteristic zero. These are used to gain information about kG and its modules (which is/are extremely complicated) from the group algebra FG, which is semisimple and therefore much better understood, via the group algebra $\mathcal{O}G$. This explains why we considered arbitrary associative rings (resp. algebras / fields) in the previous chapters rather than immediately focusing on fields of positive characteristic.

Notation. Throughout this chapter, unless otherwise specified, we let p be a prime number and let $\Lambda \in \{F, \mathcal{O}, k\}$.

15 p-Modular Systems

Recall that a commutative ring $\mathcal{O}$ is local iff $\mathcal{O}\backslash\mathcal{O}^\times = J(\mathcal{O})$, i.e. $J(\mathcal{O})$ is the unique maximal ideal of $\mathcal{O}$. Moreover, by the commutativity assumption this happens if and only if $\mathcal{O}/J(\mathcal{O})$ is a field. In such a situation, we write $k := \mathcal{O}/J(\mathcal{O})$ and call this field **the residue field of the local ring** $\mathcal{O}$. To ease up notation, we will often write $\mathfrak{p} := J(\mathcal{O})$. This is because our aim is a situation in which the residue field is a field of positive characteristic p. Moreover, a commutative ring $\mathcal{O}$ is called a **discrete valuation ring** if $\mathcal{O}$ is a local principal ideal domain such that $J(\mathcal{O}) \neq 0$. Such a discrete valuation ring is called **complete** if it is complete in the $J(\mathcal{O})$-adic topology.

Definition 15.1 (*p-modular systems*)

Let p be a prime number.

(a) A triple of rings $(F, \mathcal{O}, k)$ is called a **p-modular system** if:

   (1) $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero,

   (2) $F = \mathrm{Frac}(\mathcal{O})$ is the field of fractions of $\mathcal{O}$ (also of characteristic zero), and

   (3) $k = \mathcal{O}/J(\mathcal{O})$ is the residue field of $\mathcal{O}$ and has characteristic p.

(b) If G is a finite group, then a p-modular system $(F, \mathcal{O}, k)$ is called a **splitting p-modular system for** G, if both F and k are splitting fields for G.

It is often helpful to visualise p-modular systems and the condition on the characteristic of the rings involved through the following commutative diagram of rings and ring homomorphisms:

$$\begin{array}{ccccc} \mathbb{Q} & \hookleftarrow & \mathbb{Z} & \twoheadrightarrow & \mathbb{F}_p \\ \downarrow & & \downarrow & & \downarrow \\ F & \hookleftarrow & \mathcal{O} & \twoheadrightarrow & k \end{array}$$

where the hook arrows are the canonical inclusions and the two-head arrows the quotient morphisms. Clearly, these morphisms also extend naturally to ring homomorphisms

$$FG \hookleftarrow \mathcal{O}G \twoheadrightarrow kG$$

between the corresponding group algebras (each mapping an element $g \in G$ to itself).

Example 13

One usually works with a splitting p-modular system for all subgroups of G, because it allows us avoid problems with field extensions. By a theorem of Brauer on splitting fields such a p-modular system can always be obtained by adjoining a primitive m-th root of unity to $\mathbb{Q}_p$, where m is the exponent of G. (Notice that this extension is unique.) So we may as well assume that our situation is as given in the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{Q}_p & \hookleftarrow & \mathbb{Z}_p & \twoheadrightarrow & \mathbb{F}_p \\ \downarrow & & \downarrow & & \downarrow \\ F & \hookleftarrow & \mathcal{O} & \twoheadrightarrow & k \end{array}$$

More generally, we have the following result, which we mention without proof. The proof can be found in §17A of Volume 1 of Curtis and Reiner's book.

Theorem 15.2

Let $(F, \mathcal{O}, k)$ be a p-modular system. Let G be a finite group of exponent $m := exp(G)$. Then the following assertions hold.

(a) The field F contains all m-th roots of unity if and only if F contains the cyclotomic field of m-th roots of unity;

(b) If F contains all m-th roots of unity, then so does k and F and k are splitting fields for G and all its subgroups.

Remark 15.3

If $(F, \mathcal{O}, k)$ is a p-modular system, then it is not possible to have F and k algebraically closed, while assuming $\mathcal{O}$ is complete. (Depending on your knowledge on discrete valuation rings, you can try to prove this as an exercise!)

Let us now investigate changes of the coefficients given in the setting of a p-modular system for group algebras involved.

Definition 15.4

Let $\mathcal{O}$ be a commutative local ring. A finitely generated $\mathcal{O}G$-module L is called an **$\mathcal{O}G$-lattice** if it is free (= projective) as an $\mathcal{O}$-module.

Remark 15.5 (*Changes of the coefficients*)

Let $(F, \mathcal{O}, k)$ be a p-modular system and write $\mathfrak{p} := J(\mathcal{O})$. If L is an $\mathcal{O}G$-module, then:

- setting $L^F := F \otimes_{\mathcal{O}} L$ defines an FG-module, and
- reduction modulo $\mathfrak{p}$ of L, that is $\overline{L} := L/\mathfrak{p}L \cong k \otimes_{\mathcal{O}} L$ defines a kG-module.

We note that, when seen as an $\mathcal{O}$-module, an $\mathcal{O}G$-module L may have torsion, which is lost on passage to F. In order to avoid this issue, we usually only work with $\mathcal{O}G$-lattices. In this way, we obtain functors

$$FG\text{-}\mathbf{mod} \longleftarrow \mathcal{O}G\text{-}\mathbf{lat} \longrightarrow\!\!\!\!\!\rightarrow kG\text{-}\mathbf{mod}$$

between the corresponding categories of finitely generated $\mathcal{O}G$-lattices and finitely generated FG-, kG-modules.

A natural question to ask is: which FG-modules, respectively kG-modules, come from $\mathcal{O}G$-lattices? In the case of FG-modules we have the following answer.

Proposition-Definition 15.6

Let $\mathcal{O}$ be a complete discrete valuation ring and let $F := \mathrm{Frac}(\mathcal{O})$ be the fraction field of $\mathcal{O}$. Then, for any finitely generated FG-module V there exists an $\mathcal{O}G$-lattice L which has an $\mathcal{O}$-basis which is also an F-basis. In this situation $V \cong L^F$ and we call L an **$\mathcal{O}$-form** of V.

Proof: Choose an F-basis $\{v_1, \dots, v_n\}$ of V and set $L := \mathcal{O}Gv_1 + \cdots + \mathcal{O}Gv_n \subseteq V$. ■

On the other hand, the question has a negative answer for kG-modules.

Definition 15.7 (*liftable kG-module*)

Let $\mathcal{O}$ be a commutative local ring with unique maximal ideal $\mathfrak{p} := J(\mathcal{O})$ and residue field $k := \mathcal{O}/\mathfrak{p}$. A kG-module M is called **liftable** if there exists an $\mathcal{O}G$-lattice $\widehat{M}$ whose reduction modulo $\mathfrak{p}$ of M is isomorphic to M, that is

$$\widehat{M}/\mathfrak{p}\widehat{M} \cong M\,.$$

(Alternatively, it is also said that M is **liftable to an $\mathcal{O}G$-lattice**, or **liftable to $\mathcal{O}$**, or **liftable to characteristic zero**.)

Even though every $\mathcal{O}G$-lattice can be reduced modulo $\mathfrak{p}$ to produce a kG-module, not every kG-module is liftable to an $\mathcal{O}G$-lattice.

Being liftable for a kG-module is a rather rare property. However, some classes of kG-modules do lift.

Example 14

It follows from the *lifting of idempotents* theorem that projective indecomposable kG-modules are liftable to projective indecomposable $\mathcal{O}G$-lattices:

Any (projective) indecomposable kG-module is liftable to a (projective) indecomposable $\mathcal{O}G$-lattice. More generally, any trivial source kG-module M is liftable to an $\mathcal{O}G$-lattice. More precisely, among all lifts of M a unique one is again trivial source and we denote it by $\widetilde{M}$.
The F-character of $F \otimes_{\mathcal{O}} \widetilde{M}$ is called **the ordinary character of** M.

16 Brauer Characcters

Recall that we have fixed a splitting p-modular system $(F, \mathcal{O}, k)$ such that F contains an $\exp(G)$-th root of unity. Since F is a field of characteristic zero, FG-modules are isomorphic if and only if their characters are equal. Also, the character of an FG-module provides complete information about its composition factors, including multiplicities, provided that the irreducible characters of G are known. All this does not hold for fields k of characteristic $p > 0$. For instance, if W is a k-vector space on which G acts trivially and $\dim_k(W) = ap + 1$ for some non-negative integer a, then the k-character of W is the trivial character. This implies that a k-character can only give information about multiplicities of composition factors modulo p. In view of these issues, the aim of this chapter is to define a slightly different kind of *character theory* for modular representations of finite groups and to establish links with ordinary character theory.

Recall that an element $g \in G$ is called a ***p*-regular element** (or a p'-**element**) if $p \nmid o(g)$. We write

$$G_{p'} := \{g \in G \mid p \nmid o(g)\}$$

for the set of all p-regular elements of G.

Since F contains all $\exp(G)$-th roots of unity, both F and k contain a primitive a-th root of unity, where a is the l.c.m. of the orders of the p-regular elements. Set

$$\mu_F := \{a\text{-th roots of 1 in } F\} \text{ and } \mu_k := \{a\text{-th roots of 1 in } k\}.$$

Then $\mu_F \subseteq \mathcal{O}$ and, as both μ_F and μ_k are finite groups, it follows that the quotient morphism $\mathcal{O} \twoheadrightarrow \mathcal{O}/\mathfrak{p}$ restricted to μ_F induces a group isomorphism

$$\mu_F \xrightarrow{\cong} \mu_k .$$

We write the underlying bijection as $\widehat{\xi} \mapsto \xi$, so that if ξ is an a-th root of unity in k then $\widehat{\xi}$ is the unique a-th root of unity in $\mathcal{O}$ which maps onto it.

Lemma 16.1 (*Diagonalisation lemma*)

Let $\rho : G \longrightarrow \mathrm{GL}(U)$ be a k-representation of G. Then, for every p-regular element $g \in G_{p'}$, the k-linear map $\rho(g)$ is diagonalisable and the eigenvalues of $\rho(g)$ are $o(g)$-th roots of unity and lie in μ_k. In other words, there exists an ordered k-basis B of U with respect to which

$$\big(\rho(g)\big)_B = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi_n \end{bmatrix},$$

where $n := \dim_k(U)$ and each ξ_i $(1 \leqslant i \leqslant n)$ is an $o(g)$-th root of unity in k.

Proof: Let $g \in G_{p'}$. It is enough to consider the restriction of ρ to the cyclic subgroup $\langle g \rangle$. Since $p \nmid |\langle g \rangle|$, $k\langle g \rangle$ is semisimple by Maschke's Theorem. Moreover, as k is a splitting field for $\langle g \rangle$, it follows from Corollary 5.10 that all irreducible k-representations of $\langle g \rangle$ have degree 1. Hence $\rho|_{\langle g \rangle}$ can be decomposed as the direct sum of degree 1 subrepresentations. As a consequence $\rho(g) = \rho|_{\langle g \rangle}(g)$ is diagonalisable and there exists a k-basis B of U satisfying the statement of the lemma. It follows immediately that the eigenvalues are $o(g)$-th roots of unity because $\rho_U(g^{o(g)}) = \rho_U(1_G) = \mathrm{Id}_U$. They all lie in μ_k, being $o(g)$-th roots of unity, hence a-th roots of unity. ■

This leads to the following definition.

Definition 16.2 (*Brauer characters*)

Let U be a kG-module of dimension $n \in \mathbb{Z}_{\geqslant 0}$ and let $\rho_U : G \to \mathrm{GL}(U)$ be the associated k-representation. The p-**Brauer character** or simply the **Brauer character** of G afforded by U (resp. of ρ_U) is the F-valued function

$$\begin{aligned} \varphi_U : G_{p'} &\to \mathcal{O} \subseteq F \\ g &\mapsto \widehat{\xi_1} + \cdots + \widehat{\xi_n}\,, \end{aligned}$$

where $\xi_1, \ldots, \xi_n \in \mu_k$ are the eigenvalues of $\rho_U(g)$. The integer n is also called the **degree** of φ_U. Moreover, φ_U is called **irreducible** if U is simple (resp. if ρ_U is irreducible), and it is called **linear** if $n = 1$. We denote by $\mathrm{IBr}_p(G)$ the set of all irreducible Brauer characters of G and we write $\mathbf{1}_{G_{p'}}$ for the Brauer character of the trivial kG-module.

In the sequel, we want to prove that Brauer characters of kG-modules have properties similar to $\mathbb{C}$-characters.

Remark 16.3

(a) **Warning**: $\varphi(g) \in \mathcal{O} \subseteq F$ even though $\rho_U(g)$ is defined over the field k of characteristic $p > 0$.

(b) Often the values of Brauer characters are considered as complex numbers, i.e. sums of complex roots of unity. Of course, in that case then $\varphi_U(g)$ depends on the choice of embedding of μ_F into $\mathbb{C}$. However, for a fixed embedding, $\varphi_U(g)$ is uniquely determined up to similarity of $\rho_U(g)$.

Immediate properties of Brauer characters are as follows.

Exercise 16.4

Let U, V, W be non-zero kG-modules. Prove the following assertions:

(a) $\varphi_U(1) = \dim_k(U)$.

(b) φ_U is a class function on $G_{p'}$.

(c) $\varphi_U(g^{-1}) = \varphi_{U*}(g)$ $\forall\ g \in G_{p'}$.

(d) If $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is a s.e.s. of kG-modules, then

$$\varphi_V = \varphi_U + \varphi_W \,.$$

(e) If the composition factors of U are $S_1, \ldots, S_m$ ($m \in \mathbb{Z}_{\geqslant 1}$) with multiplicities $n_1, \ldots, n_m$ respectively, then

$$\varphi_U = n_1\varphi_{S_1} + \ldots + n_m\varphi_{S_m} \,.$$

In particular, if two kG-modules have isomorphic composition factors, counting multiplicities, then they have the same Brauer character.

(f) $\varphi_{U\oplus V} = \varphi_U + \varphi_V$ and $\varphi_{U\otimes_k V} = \varphi_U \cdot \varphi_V \,.$

(g) Assume U is a liftable and $\widehat{U}$ is a lift, i.e. $\widehat{U}/\mathfrak{p}\widehat{U} \cong U$. Write $\chi_{\widehat{U}}$ be the F-character of $F \otimes_{\mathcal{O}} \widehat{U}$. Then $\varphi_U(g) = \chi_{\widehat{U}}(g)$ on all p-regular elements $g \in G$.

Brauer proved that Brauer characters can be counted using conjugacy classes as well:

Theorem 16.5

The set $\mathrm{IBr}_p(G)$ of irreducible Brauer characters of G forms an F-basis of the F-vector space $\mathrm{Cl}_F(G_{p'})$ of class functions on $G_{p'}$ and

$$|\,\mathrm{IBr}_p(G)| = \dim_F \mathrm{Cl}_F(G_{p'}) = \text{number of conjugacy classes of } p\text{-regular elements in } G\,.$$

We note that the second equality is obvious, because the indicator functions on the conjugacy classes of p-regular elements form an F-basis.

17 Back to reduction modulo $\mathfrak{p}$

We now want to investigate the connections between representations of G over F (or $\mathbb{C}$) and representations of G over k through the connections between their F-characters and Brauer characters.

Proposition 17.1

Let V be an FG-module with F-character χ_V. Then:

(a) there exists an $\mathcal{O}G$-lattice L such that $V \cong F \otimes_{\mathcal{O}} L$ (called an **$\mathcal{O}$-form of V**);

(b) $\chi_V|_{G_{p'}} = \varphi_{\overline{L}}$ and is called the **reduction modulo p of χ_V**;

(c) if $V \in \mathrm{Irr}_F(G)$, there exist non-negative integers $d_{\chi\varphi}$ such that

$$\chi_V|_{G_{p'}} = \sum_{\varphi\in \mathrm{IBr}_p(G)} d_{\chi\varphi}\varphi.$$

Exercise 17.2

Assume G is a p-group. Prove that the reduction modulo p of any linear character is the trivial Brauer character.

Definition 17.3

The matrix

- $D := \mathrm{Dec}_p(G) = (d_{\chi\varphi})_{\substack{\chi\in\mathrm{Irr}_F(G)\\ \varphi\in\mathrm{IBr}_p(G)}}$ is the **p-decomposition matrix** of G;
- $C := D^{tr}D = (c_{\varphi\mu})_{\varphi,\mu\in\mathrm{IBr}_p(G)}$ is the **Cartan matrix** of G.

Proposition 17.4

(a) The decomposition matrix $\mathrm{Dec}_p(G)$ has full rank, namely $|\mathrm{IBr}_p(G)|$.

(b) The Cartan matrix of G is a symmetric positive definite matrix with non-negative integer entries.

Recall now that projective kG-modules are liftable and this enables us to associate an F-character of G to each PIM of kG, in fact in a unique way in this case.

Definition 17.5

Let $\varphi \in \mathrm{IBr}_p(G)$ be an irreducible Brauer character afforded by a simple kG-module S. Let P_S be the projective cover of S and let $\widehat{P}_S$ denote a lift of P_S to $\mathcal{O}$. Then, the F-character of $(\widehat{P}_S)^F$ is denoted by Φ_φ and is called the **projective indecomposable character** associated to S or φ.

Proposition 17.6

Let $\varphi \in \mathrm{IBr}_p(G)$. Then:

(a) $\Phi_\varphi = \sum_{\chi\in\mathrm{Irr}_F(G)} d_{\chi\varphi}\chi$; and

(b) $\Phi_\varphi|_{G_{p'}} = \sum_{\mu\in\mathrm{IBr}_p(G)} c_{\varphi\mu}\mu$.

Definition 17.7 (*Brauer character table*)

Set $l := |G_{p'}|$ and let $g_1, \ldots, g_l$ be a complete set of representatives of the p-regular conjugacy classes of G.

(a) The **Brauer character table** of a finite group G is the matrix $\Big(\varphi(g_j)\Big)_{\substack{\varphi\in\mathrm{IBr}_p(G)\\ 1\leqslant j\leqslant l}} \in M_l(F)$.

(b) The **Brauer projective table** of a finite group G at p is the matrix $\Big(\Phi_\varphi(g_j)\Big)_{\substack{\varphi\in\mathrm{IBr}_p(G)\\ 1\leqslant j\leqslant l}} \in M_l(F)$.

Wednesday. Chapter 5. Block Theory

We now want to break down the representation theory of finite groups into its smallest parts: the *blocks* of the group algebra. Before we proceed, I want to give the following warning: one of the confusing things about the block theory of finite groups is that there often seems to be more than one definition of the same concept. In fact several different definitions – and mathematical objects – are hidden behind the word *block* of a group algebra. Some texts consider blocks to be *algebras*, or more precisely indecomposable 2-sided ideals of the group algebra, some to be *primitive central idempotents* of the group algebra, some to be the union of the sets of irreducible ordinary characters and irreducible Brauer characters of the aforementioned algebra, some others to be an equivalence class of modules over the group algebra (sometimes simple, sometimes indecomposable, sometimes arbitrary),... Important is to keep in mind, that although different authors use different approaches, there are essentially equivalent. We will focus here on the algebra approach.

Notation: We keep the notation and the assumptions of the previous Chapters. Throughout, G denotes a finite group, p a prime number. We let $(F, \mathcal{O}, k)$ denote a p-modular system and we assume F contains all $\exp(G)$-th roots of unity, so $(F, \mathcal{O}, k)$ is a splitting p-modular system for G and all its subgroups (see Theorem 15.2). We write $\mathfrak{p} := J(O)$ and we let $\Lambda \in \{F, \mathcal{O}, k\}$.

18 The p-Blocks of a Group

The *block decomposition* of the group algebra ΛG is just the the decomposition of ΛG, seen a $(\Lambda G, \Lambda G)$-bimodule, into indecomposable $(\Lambda G, \Lambda G)$-bimodules. So in block theory of finite groups, by definition, one should work with bimodules. However, bimodules over group algebras can always be made into one-sided modules as described in the following remark.

Remark 18.1

Let G_1 and G_2 be finite groups. If M is a $(\Lambda G_1, \Lambda G_2)$-bimodule, then M can be endowed with the structure of a one-sided $\Lambda[G_1 \times G_2]$-module via the $G_1 \times G_2$-action:

$$\cdot : (G_1 \times G_2) \times M \longrightarrow M,\ m \mapsto (g, h) \cdot m := g \cdot m \cdot h^{-1}$$

The consequence is that the one-sided module theoretic terms and results we have seen so far can be applied to such bimodules. Thus, in the sequel, we identify bimodules with one-sided left modules without further mention.

Definition 18.2 (***Blocks of the group algebra***)

In the unique decomposition $\Lambda G = B_1 \oplus \cdots \oplus B_n$ into indecomposable $(\Lambda G, \Lambda G)$-subbimodules of ΛG, the summands $B_1, \ldots, B_n$ are called the **blocks** of ΛG. (Or sometimes just **block algebras**.)

Remark 18.3

The block decomposition $\Lambda G = B_1 \oplus \cdots \oplus B_n$ is equivalent to a decomposition

$$1 = \underbrace{e_1}_{\in B_1} + \ldots + \underbrace{e_n}_{\in B_n}$$

of the unit of ΛG as a sum of orthogonal primitive central idempotents $e_i \in Z(\Lambda G)$, where $e_i = 1_{B_i}$ and $B_i = \Lambda G e_i \, \forall \, 1 \leqslant i \leqslant n$. We call the elements $e_1, \ldots, e_n$ the **block idempotents** of ΛG.

Definition 18.4 (***belonging to a block***)

We say that an (indecomposable) ΛG-module M **belongs to (or lies in) the block** $B_i = \Lambda G e_i$ if $e_i M = M$ and $e_j M = 0$ for all $1 \leqslant j \leqslant n$ such that $j \neq i$.

Remark 18.5

It follows from the previous remark, that every *indecomposable* ΛG-module M belongs to a uniquely determined block of ΛG. Indeed, the decomposition

$$1 = e_1 + \ldots + e_n \implies M = 1 \cdot M = e_1 \cdot M \oplus \ldots \oplus e_n \cdot M$$

but as M is indecomposable the Krull-Schmidt theorem tells us that

$$\exists!\, 1 \leqslant i \leqslant n \text{ such that } e_i M = M \text{ and } e_j M = 0 \;\forall\, 1 \leqslant j \leqslant n \text{ with } j \neq i\,.$$

Definition 18.6 (***Principal block***)

The principal block of ΛG is the block to which the trivial module Λ belongs. **Notation**: $B_0(\Lambda G)$.

Exercise 18.7

(a) Let B_i be a block of ΛG and let e_i be the corresponding block idempotent. Prove that a ΛG-module M belongs to B_i if and only if external multiplication by e_i is a ΛG-isomorphism on that module.

(b) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of ΛG-modules and ΛG-homomorphisms. Prove that, for each $1 \leqslant i \leqslant n$:

$$M \text{ belong to the block } B_i \text{ of } \Lambda G \text{ if and only } L \text{ and } N \text{ belong to } B_i\,.$$

[Hint: use (a) and the 5-Lemma.]

(c) Deduce that if a ΛG-module M lies in a block B of ΛG, then so do all of its submodules and all of its factor modules.

Example 15 (*Blocks of FG*)

Since FG is semisimple, the block decomposition of FG is given by the Artin-Wedderburn Theorem. In particular, the blocks are matrix algebras and can be labelled by $\mathrm{Irr}(FG)$. (Or $\mathrm{Irr}_F(G)$ if you prefer!)

Remark 18.8 (*Blocks of $\mathcal{O}G$ and kG*)

The Lifting of Idempotents tells us that the quotient morphism $\mathcal{O}G \twoheadrightarrow [\mathcal{O}/\mathfrak{p}]G = kG, x \mapsto \overline{x}$ induces a bijection

$$\begin{array}{ccc} \{\text{primitive idempotents of } Z(\mathcal{O}G)\} & \xleftrightarrow{\sim} & \{\text{primitive idempotents of } Z(kG)\} \\ e & \mapsto & \overline{e}\ . \end{array}$$

Thus a decomposition $1_{\mathcal{O}G} = e_1 + \cdots + e_r$ of the identity element of $\mathcal{O}G$ into a sum of primitive central idempotents corresponds to a decomposition $1_{kG} = \overline{e}_1 + \cdots + \overline{e}_r$ of the identity element of kG into a sum of primitive central idempotents of kG. Therefore, by Proposition 18.3, there is a bijection between the blocks of $\mathcal{O}G$ and the blocks of kG:

$$\begin{array}{ccccccccc} \mathcal{O}G & = & B_1 & \oplus & \cdots & \oplus & B_n \\ \downarrow\!\!\downarrow & & \updownarrow & & \cdots & & \updownarrow \\ kG & = & \overline{B_1} & \oplus & \cdots & \oplus & \overline{B_n} \end{array}$$

We define a p-**block** of G to be the specification of a block of $\mathcal{O}G$, understanding also the corresponding block of kG. We write $\mathrm{Bl}_p(G)$ for the set of all p-blocks of G when it is clear from the context/unimportant whether we work over $\mathcal{O}$ or over k, resp. $\mathrm{Bl}_p(\mathcal{O}G)$ for the set of all blocks of $\mathcal{O}G$ and $\mathrm{Bl}_p(kG)$ for the set of all blocks of kG.

The division of the simple kG-modules into blocks can be achieved in a purely combinatorial fashion, knowing the Cartan matrix of kG. The connection with a block matrix decomposition of the Cartan matrix is probably the origin of the use of the term *block* in representation theory.

Remark 18.9

On listing the simple kG-modules so that modules in each block occur together, the Cartan matrix of kG has a block diagonal form, with one block matrix for each p-block of the group. Up to permutation of simple modules within p-blocks and permutation of the p-blocks, this is the unique decomposition of the Cartan matrix into block diagonal form with the maximum number of block matrices.

19 Defect Groups

From now on we will only discuss the blocks of kG. (Analogous results hold for the corresponding blocks of $\mathcal{O}G$.) We write $\Delta : G \longrightarrow G \times G, g \mapsto (g, g)$ for the diagonal embedding of G in $G \times G$.

We start with a result, which lets us identify the vertices of a block with a conjugacy class of p-subgroups of G.

Theorem 19.1

If $B \in \mathrm{Bl}_p(kG)$, then every vertex of B, considered as an indecomposable $k[G \times G]$-module, has the form $\Delta(D)$ for some p-subgroup $D \leqslant G$. Moreover, D is uniquely determined up to conjugation in G.

Definition 19.2 (*Defect group, defect*)

Let $B \in \mathrm{Bl}_p(kG)$.

(a) A **defect group** of B is a p-subgroup $D \leqslant G$ such that $\Delta(D)$ is a vertex of B considered as an indecomposable $k[G \times G]$-module.

(b) If $|D| = p^d$ ($d \in \mathbb{Z}_{\geqslant 0}$) then d is called the **defect** of B.

Note. As the vertices of a module form a a conjugacy class of subgroups, so do the defect groups of a block and it is clear that in fact all defect groups have the same order.

Defect groups are useful and important because in some sense they measure how far a p-block is from being semisimple (see Exercise 19.5 below). In general they are very difficult to determine concretely. However, the following properties (mostly due to Green) are useful.

Properties 19.3

Let $B \in \mathrm{Bl}_p(kG)$ with defect group $D \leqslant G$. Then the following assertions hold.

(a) If B is a block of kG with defect group D, then every indecomposable kG-module belonging to B is relatively D-projective, and hence has a vertex contained in D.

(b) D contains every normal p-subgroup of G;

(c) D is the largest normal p-subgroup of $N_G(D)$, i.e. $Q = O_p(N_G(Q))$.

Example 16

Since the vertices of the trivial kG-module k are the Sylow p-subgroups of G, so are the defect groups of the principal group $B_0(kG)$.

Exercise 19.4 (*p-block(s) of a p-group*)

Prove that if G is a p-group, then G has a unique p-block.

Exercise 19.5

Let B be a block of kG with a trivial defect group. Prove that B is a semisimple algebra.

Finally we present a fundamental result due to Brauer.

Definition 19.6

Let $H \leqslant G$, let $b \in \mathrm{Bl}_p(kH)$. Then a block $B \in \mathrm{Bl}_p(kG)$ **corresponds to** b if and only if $b \mid B{\downarrow}^{G\times G}_{H\times H}$ *and* B is the unique block of kG with this property. We then write $B = b^G$. If such a block B exists, then we say that b^G **is defined**.

Theorem 19.7 (*Brauer's First Main Theorem*)

Let $D \leqslant G$ be a p-subgroup and let $H \leqslant G$ containing $N_G(D)$. Then, there is a bijection

$$\{\text{Blocks of } kH \text{ with defect group } D\} \xrightarrow{\sim} \{\text{Blocks of } kG \text{ with defect group } D\}$$
$$b \mapsto b^G$$

Moreover, in this case b^G is called the **Brauer correspondent** of b (and conversely).

Proof (Sketch): This is a particular case of the Green correspondence (i.e. when viewing blocks as one-sided left modules). ■

Many of the results and open problems in modular representation theory of finite groups are concerned with the influence of the structure of the defect group on the structure of the block. For example, by a result of Brauer, $|D|$ is the largest elementary divisor of the Cartan matrix of a block B with defect group D, and it appears with multiplicity 1. We mention here two major open problems in this spirit.

Conjecture 19.8 (*Brauer's* $\mathtt{k}(B)$*-Conjecture*)

Let $B \in \mathrm{Bl}_p(kG)$ with defect group D. Then $|\operatorname{Irr}_F(B)| \leqslant |D|$.

Conjecture 19.9 (*Broué's Abelian Defect Group Conjecture*)

Let $B \in \mathrm{Bl}_p(G)$ with abelian defect group D and let $b \in \mathrm{Bl}_p(N_G(D))$ be the Brauer correspondent of B. Then, the derived categories $D^b(\mathrm{mod}(B))$ and $D^b(\mathrm{mod}(b))$ of bounded complexes of finitely generated modules over B and b are equivalent as triangulated categories.

20 Equivalences of Block Algebras

Basic Question 20.1 (*Open!!*)

Which k-algebras (resp. $\mathcal{O}$-algebras) occur as p-blocks of finite groups?

Conjectural Answer 20.2

If a defect group is fixed, only finitely many ... up to a *good* notion of equivalence!

In this respect, Donovan's and Puig's Conjectures are further good examples of open problems concerned with the influence of the structure of the defect group on the structure of the block.

Conjecture 20.3 (*Donovan's/Puig's Conjecture, '70's/'80's*)

Let D be a finite p-group. Then, there exists only finitely many (splendid) Morita equivalence classes of p-blocks of finite groups with a defect group isomorphic to D.

Donovan's Conjuecture is known to hold over $\mathcal{O}$ and over k for a fairly long list of *small* defect groups. The status of this conjecture is kept up-to-date by Charles Eaton on the Wiki page of his *block library*. See `https://wiki.manchester.ac.uk/blocks/index.php/Main_Page`.

On the other hand, not much is known towards Puig's Conjecture. It is known to hold if $D \cong C_{p^n}$, that is, is a cyclic p-group (Linckelmann, 1996) and if $D \cong C_2 \times C_2$ if $p = 2$ (Craven-Eaton-Kessar-Linckelmann, 2012).

Here:

Definition 20.4 (*Morita equivalence*)

Let G and G' be two finite groups. Two block algebras $A \in \mathrm{Bl}_p(G)$ and $B \in \mathrm{Bl}_p(G')$ are called **Morita equivalent** iff $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ are equivalent as (k-linear, resp. $\mathcal{O}$-linear) categories. If this is the case, then we write $A \sim_M B$.

The following result on Morita equivalences is often useful in order to verify that such an equivalence exists.

Theorem 20.5 (*Morita's Theorem*)

With the assumptions and notation of the previous definition, TFAE:

(a) $A \sim_M B$; and

(b) there exists an (A, B)-bimodule M and a (B, A)-bimodule N such that $M \otimes_B N \cong A$ (as (A, A)-bimodules) and $N \otimes_A M \cong B$ (as (B, B)-bimodules).

In fact in the case of block algebras, N is the dual of M. Therefore, we often say that the Morita equivalence is **induced** of **realised** by the bimodule M of Assertion (b) of Morita's Theorem.

Definition 20.6 (*Morita equivalence*)

Let G and G' be two finite groups. Assume M is an (A, B)-bimodule realising a Morita equivalence bewtween $A \in \mathrm{Bl}_p(kG)$ and $B \in \mathrm{Bl}_p(kG')$. This Morita equivalence is called:

- a **splendid Morita equivalence** (or also a **source-algebra equivalence** or a **Puig equivalence**) iff the bimodule M, seen as a left $k[G \times G']$-module, is a p-permutation module, and if it is the case we write $A \sim_{SM} B$
- an **endo-permutation source equivalence** (or also a **basic equivalence**) iff the bimodule M, seen as a left $k[G \times G']$-module, has a source T such that $\mathrm{End}_k(T) \cong$ permutation module.

Morita and splendid Morita equivalences of occur naturally in the modular representation theory of finite groups. Standard examples are as follows:

Example 17 (***Examples of (splendid) Morita equivalences in the block theory of finite groups***)

(a) Isomorphic blocks (i.e. as k-algebras) are always Morita equivalent.

(b) $B_0(kG) \sim_{SM} B_0(k[G/O_{p'}(G)])$ because $O_{p'}(G)$ always acts trivially on the principal block.

(c) **"Alperin/Dade"**. If $G \trianglelefteq \widetilde{G}$ and there is a Sylow p-subgroup P of G such that $\widetilde{G} = GC_{\widetilde{G}}(P)$, then

$$B_0(k\widetilde{G}) \sim_{SM} B_0(kG)\,.$$

In fact in this case the two principal blocks are isomorphic.

(d) **"Fong-Reynolds"**. If $H \trianglelefteq G$, $b \in \mathrm{Bl}_p(H)$, $T := \mathrm{Stab}_G(b)$, then there exists a bijection

$$\mathrm{Bl}_p(T \mid b) \xrightarrow{\sim} \mathrm{Bl}_p(B \mid b), B \mapsto B^G$$

where the bimodule $M := 1_{B^G} \cdot kG \cdot 1_B$ realises a splendid Morita equivalence between B and B^G.

Remark 20.7

It can be proved that splendidly Morita equivalent blocks and basically equivalent blocks necessarily have isomorphic defect groups.
Whether Morita equivalent blocks necessarily have isomorphic defect groups was an open question for a long time. However, as mentioned by Claudio in his talk, a special case is the *modular isomorphism problem*, which has recently (July 2021) been shown to have a negative answer by Garcia-Margolis-Del Rio. More precisely, they prove that there are non-isomorphic finite 2-groups G and G' such that the group rings of G and G' over any field of characteristic 2 are isomorphic.

Finally we mention that the notions of a Morita and a splendid Morita equivalence can be weakened in different flavours to equivalences between the stable module categories or of the bounded derived categories of the blocks.

Definition 20.8 (***Rickard equivalence / Stable equivalence of Morita type***)

Let G and G' be two finite groups. Two block algebras $A \in \mathrm{Bl}_p(G)$ and $B \in \mathrm{Bl}_p(G')$ are called:

(a) **Rickard** (or **derived**) **equivalent** if the derived categories $D^b(\mathrm{mod}(A))$ and $D^b(\mathrm{mod}(B))$ of bounded complexes of finitely generated modules over A and B are equivalent as triangulated categories.

(b) "**stably equivalent à la Morita**" (or say that there is a **stable equivalence of Morita type** between A and B) if there exist an (A, B)-bimodule M which is projective as an A-module and as a B-module and a (B, A)-bimodule N which is projective as a B-module and as an A-module such that $M \otimes_B N \cong A \oplus$ (projectives) as (A, A)-bimodules and $N \otimes_A M \cong B \oplus$ (projectives) as (B, B)-bimodules.

Remark 20.9

(a) A derived version of Morita's theorem asserts that A and B are Rickard equivalent if and only if the equivalence of triangulated categories between $D^b(\mathrm{mod}(A))$ and $D^b(\mathrm{mod}(B))$ can be realised by tensoring over B with a bounded complex $M_\bullet$ of (A, B)-bimodules in which each term is both projective as an A-module and as B-module. When all terms in this complex (seen as one-sided left modules) are p-permutation modules, then the equivalence is called a **splendid Rickard equivalence**.

(b) A stable equivalence of Morita type between A and B induces an equivalence of triangulated categories between the stable categories $\mathrm{stmod}(A)$ and $\mathrm{stmod}(B)$.

See the HANDOUT of my Beamer presentation for relations between these equivalences.

Finally, we mention that blocks with cyclic defect groups are a very nice playground to play around with several notions of equivalences – mentioned in this section – and more concepts such as the Green correspondence, Clifford theory or perfect isometries of characters.

Remark 20.10 (*Blocks with cyclic defect groups*)

Let $B \in \mathrm{Bl}_p(kG)$ be a block with a cyclic defect group D. Let D_1 be the unique cyclic subgroup of D of order p. As D is cyclic, $N_1 := N_G(D_1) \geqslant N_G(D)$, so we may consider the Brauer correspondent $b \in \mathrm{Bl}_p(N_G(D_1))$ of B, let $c \in \mathrm{Bl}_p(C_G(D_1))$ be a block of $C_G(D_1)$ covered by b and let $b' \in \mathrm{Bl}_p(\mathrm{Stab}_{N_G(D_1)}(c))$ be the Fong-Reynolds correspondent of c. Then, we have the following situation: