## Tuesday. Chapter 3. Indecomposable Modules

After simple and semisimple modules, the goal of this chapter is to understand indecomposable modules in general. Apart for exceptions, the group algebra is of *wild representation type*, which, roughly speaking, means that it is not possible to classify the indecomposable modules over such algebras. However, representation theorists have developed tools which enable us to organise indecomposable modules in packages parametrised by parameters that are useful enough to understand essential properties of these modules. In this respect, we will generalise the idea of a projective module by defining what is called **relative projectivity**. This will lead us to introduce the concepts of **vertices** and **sources** of indecomposable modules, which are two typical examples of parameters bringing us useful information about indecomposable modules in general.

## 8 Existence and Uniqueness of Direct Sum Decompositions

First, we take a look at the concept of decomposability over general rings.

**Definition 8.1 (*indecomposable module*)**

An $R$-module $M$ is called **decomposable** if $M$ possesses two non-zero proper submodules $M_1, M_2$ such that $M = M_1 \oplus M_2$. An $R$-module $M$ is called **indecomposable** if it is non-zero and not decomposable.

First, we want to be able to decompose $R$-modules into direct sums of indecomposable submodules. The Krull-Schmidt Theorem then provide us with certain uniqueness properties of such decompositions.

**Proposition 8.2**

Let $M$ be an $R$-module. If $M$ satisfies either A.C.C. or D.C.C., then $M$ admits a decomposition into a direct sum of finitely many indecomposable $R$-submodules.

**Exercise 8.3**

Prove Proposition 8.2.

**Theorem 8.4 (*Krull–Schmidt*)**

Let $M$ be an $R$-module which has a composition series. If

$$M = M_1 \oplus \cdots \oplus M_n = M'_1 \oplus \cdots \oplus M'_{n'} \qquad (n, n' \in \mathbb{Z}_{>0})$$

are two decomposition of $M$ into direct sums of finitely many indecomposable $R$-submodules, then $n = n'$, and there exists a permutation $\pi \in \mathfrak{S}_n$ such that $M_i \cong M'_{\pi(i)}$ for each $1 \leqslant i \leqslant n$.

Thus the number $n$ is uniquely determined by the module $M$, and the submodules $M_1, \ldots, M_n$ are unique, up to isomorphism and ordering. They are sometimes called the **components** of $M$.

# 9 Indecomposability Criteria

The proof of the Krull-Schmidt theorem relies on the following general indecomposability criterion.

**Proposition 9.1 (*Indecomposability criterion*)**

Let $M$ be an $R$-module which has a composition series. Then:

$$M \text{ is indecomposable} \iff \mathrm{End}_R(M) \text{ is a local ring.}$$

For modules over the group algebra, we have the following important indecomposability criterion due to J. A. Green. The proof is rather involved.

**Theorem 9.2 (*Green's indecomposability criterion, 1959*)**

Assume that $K$ is an algebraically closed field of characteristic $p > 0$. Let $H \leqslant G$ be a subnormal subgroup of $G$ of index a power of $p$ and let $M$ be an indecomposable $KH$-module. Then $M\uparrow_H^G$ is an indecomposable $KG$-module.

**Remark 9.3**

Green's indecomposability criterion remains true over an arbitrary field of characteristic $p$, provided we replace *indecomposability* with *absolute indecomposability*. (A $KG$-module $M$ is called absolutely indecomposable iff its endomorphism algebra $\mathrm{End}_{KG}(M)$ is a *split local algebra*, that is, if $\mathrm{End}_{KG}(M)/J(\mathrm{End}_{KG}(M)) \cong K$.)

**Example 5**

Assume that $K$ is an algebraically closed field of characteristic $p > 0$. If $P$ is a $p$-group, $Q \leqslant P$ and $M$ is an indecomposable $KQ$-module, then $M\uparrow_Q^P$ is an indecomposable $KP$-module. In particular, the permutation module $K[P/Q] \cong K\uparrow_Q^P$ is indecomposable.

Indeed, since $P$ is a $p$-group, by the Sylow theory any subgroup $Q \leqslant P$ can be plugged in a subnormal series where each quotient is cyclic of order $p$, hence is a subnormal subgroup of $P$. The claim follows immediately from Green's indecomposability criterion.

# 10 Projective Modules for the Group Algebra

We have seen that over a semisimple ring, all simple modules appear as direct summands of the regular module with multiplicity equal to their dimension. For non-semisimple rings this is not true any more, but replacing simple modules by the *projective* modules, we will obtain a similar characterisation.

To begin with we review a series of properties of projective $KG$-modules with respect to the operations on groups and modules we have introduced in Chapter 1, i.e. induction/restriction, tensor products, ...

**Proposition 10.1**

Assume $K$ is an arbitrary commutative ring. Then the following assertions hold.

(a) If $P$ is a projective $KG$-module and $M$ is an arbitrary $KG$-module, then $P \otimes_K M$ is projective.

(b) If $P$ is a projective $KG$-module and $H \leqslant G$, then $P\downarrow_H^G$ is a projective $KH$-module.

(c) If $H \leqslant G$, then $KH\uparrow_H^G \cong KG$ and $P$ is a projective $KH$-module, then $P\uparrow_H^G$ is a projective $KG$-module. [Hint: Prove that $KH\uparrow_H^G \cong KG$.]

**Exercise 10.2**

Prove Proposition 10.1.

We now want to prove that the PIMs of $KG$ can be labelled by the simple $KG$-modules, and hence that there are a finite number of them, up to isomorphism. We will then be able to deduce from this bijection that each of them occurs in the decomposition of the regular module with multiplicity equal to the dimension of the corresponding simple module.

**Theorem 10.3**

(a) If $P$ is a projective indecomposable $KG$-module, then $P/\operatorname{rad}(P)$ is a simple $KG$-module.

(b) If $M$ is a $KG$-module and $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ for a projective indecomposable $KG$-module $P$, then there exists a surjective $KG$-homomorphism $\varphi : P \longrightarrow M$. In particular, if $M$ is also projective indecomposable, then $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ if and only if $M \cong P$.

(c) There is a bijection

$$\begin{array}{ccc} \left\{\begin{matrix}\text{projective indecomposable}\\ KG\text{-modules}\end{matrix}\right\}/\cong & \xleftrightarrow{\sim} & \left\{\begin{matrix}\text{simple}\\ KG\text{-modules}\end{matrix}\right\}/\cong \\ P & \mapsto & P/\operatorname{rad}(P) \end{array}$$

and hence the number of pairwise non-isomorphic PIMs of $KG$ is finite.

**Definition 10.4 (*Projective cover of a simple module*)**

If $S$ is a simple $KG$-module, then we denote by $P_S$ the projective indecomposable $KG$-module corresponding to $S$ through the bijection of Theorem 10.3(c) and call this module the **projective cover** of $S$.

**Corollary 10.5**

In the decomposition of the regular module $KG$ into a direct sum of indecomposable $KG$-submodules, each isomorphism type of projective indecomposable $KG$-module occurs with multiplicity $\dim_K(P/\operatorname{rad}(P))$. In other words,

$$KG \cong \bigoplus_{S\in\operatorname{Irr}(KG)} (P_S)^{n_s}$$

(where $n_S = \dim_K S$).

**Proof:** Let $KG = P_1 \oplus \cdots \oplus P_r$ ($r \in \mathbb{Z}_{>0}$) be such a decomposition. In particular, $P_1, \ldots P_r$ are PIMs. Then

$$J(KG) = J(KG)KG = J(KG)P_1 \oplus \cdots \oplus J(KG)P_r = \operatorname{rad}(P_1) \oplus \cdots \oplus \operatorname{rad}(P_r)\,.$$

Therefore,

$$KG/J(KG) \cong P_1/\operatorname{rad}(P_1) \oplus \cdots \oplus P_r/\operatorname{rad}(P_r)$$

where each summand is simple by Theorem 10.3(a). Now as $KG/J(KG)$ is semisimple, by Theorem 5.5, any simple $KG/J(KG)$-module occurs in this decomposition with multiplicity equal to its $K$-dimension. Thus the claim follows from the bijection of Theorem 10.3(c). ■

The Theorem also leads us to the following important dimensional restriction on projective modules, which we will see again later.

**Exercise 10.6**

Assume $K$ is a splitting field for $G$ of characteristic $p > 0$.

(a) Prove that if $G$ is a $p$-group, then the projective cover of the trivial module is the regular module.

(b) Use (a) and restriction to a Sylow $p$-subgroup to prove that if $P$ is a projective $KG$-module, then

$$|G|_p \mid \dim_K(P)\,.$$

(Here $|G|_p$ is the $p$-part of $|G|$, i.e. the exact power of $p$ that divides the order of $G$.)

# 11 Relative Projectivity

**Definition 11.1**

Let $H \leqslant G$. A $KG$-module $M$ is called **relatively $H$-projective**, or simply **$H$-projective**, if it is isomorphic to a direct summand of a $KG$-module induced from $H$, i.e. if $M \mid V\uparrow_H^G$ for some $KH$-module $V$.

**Example 6**

Clearly, $H$-projectivity is a generalisation of projectivity. Indeed, if $M \in \operatorname{mod}(KG)$, then:

$$\begin{aligned} M \text{ is projective} &\iff \exists n \in \mathbb{Z}_{>0} \text{ such that } M \mid (KG)^n \cong (K\uparrow_{\{1\}}^G)^n \cong (K^n)\uparrow_{\{1\}}^G \\ &\iff M \text{ is } \{1\}\text{-projective} \end{aligned}$$

We can actually characterise relative projectivity in a similar way as we characterised projectivity.

**Proposition 11.2 (*Characterisation of relative projectivity*)**

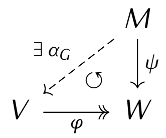
Let $H \leqslant G$ and let $M$ be a $KG$-module. TFAE:

(a) $M$ is relatively $H$-projective;

(b) $M \mid M\downarrow_H^G\uparrow_H^G$;

(c) $\exists$ a $KG$-module $N$ such that $M \mid K\uparrow_H^G \otimes_K N$;

(d) if $\psi \in \mathrm{Hom}_{KG}(M, W)$, $\varphi \in \mathrm{Hom}_{KG}(V, W)$ is surjective and $\exists\ \alpha_H \in \mathrm{Hom}_{KH}(M\downarrow_H^G, V\downarrow_H^G)$ such that $\varphi \circ \alpha_H = \psi$, then $\exists\ \alpha_G \in \mathrm{Hom}_{KG}(M, V)$ such that $\varphi \circ \alpha_G = \psi$;




(e) A surjective $KG$-homomorphism $\varphi : V \twoheadrightarrow M$ is $KG$-splits provided it is $KH$-split.

Projectivity relative to a subgroup can be generalised as follows to projectivity relative to a $KG$-module:

**Remark 11.3 (*Projectivity relative to KG-modules*)**

(a) Let $V$ be a $KG$-module. A $KG$-module $M$ is termed *projective relative to the module* $V$ or *relatively* $V$*-projective*, or simply $V$*-projective* if there exists a $KG$-module $N$ such that $M$ is isomorphic to a direct summand of $V \otimes_K N$, i.e. $M \mid V \otimes_K N$.

(b) Proposition 11.2(c) shows that projectivity relative to a subgroup $H \leqslant G$ is in fact projectivity relative to the $KG$-module $V := K\uparrow_H^G$.

The concept of projectivity relative to a subgroup is proper to the group algebra, but the concept of projectivity relative to a module is not and makes sense in general over algebras/rings.

Next we see that any indecomposable $KG$-module can be seen as a relatively projective module with respect to some subgroup of $G$.

**Theorem 11.4**

Let $H \leqslant G$.

(a) If $|G : H|$ is invertible in $K$, then every $KG$-module is $H$-projective.

(b) In particular, if $K$ is a field of characteristic $p > 0$ and $H$ contains a Sylow $p$-subgroup of $G$, then every $KG$-module is $H$-projective.

Part (b) follows immediately from (a). Indeed, if $P \in \mathrm{Syl}_p(G)$ and $H \supseteq P$, then $p \nmid |G : H|$, so $|G : H| \in K^\times$. Moreover, considering the case $H = \{1\}$ shows that Theorem 11.4 is a generalisation of Maschke's Theorem.

**Example 7**

Assume that $\text{char}(K) =: p > 0$ and $H = \{1\}$. If $H$ contains a Sylow $p$-subgroup of $G$ then the Sylow $p$-subgroups of $G$ are trivial, so $p \nmid |G|$. The theorem then says that all $KG$-modules are $\{1\}$-projective, that is, projective.

We know this already, however! If $p \nmid |G|$ then $KG$ is semisimple by Maschke's Theorem, and so all $KG$-modules are projective.

**Corollary 11.5**

Let $H \leqslant G$ and suppose that $|G : H|$ is invertible in $K$. Then a $KG$-module $M$ is projective if and only if $M{\downarrow}_H^G$ is projective.

**Proof:** The necessary condition is given by Proposition 10.1(b). To prove the sufficient condition, suppose that $M{\downarrow}_H^G$ is projective. Then, on the one hand,

$$M{\downarrow}_H^G \mid (KH)^n \qquad \text{for some } n \in \mathbb{Z}_{>0} .$$

On the other hand, $M$ is $H$-projective by Theorem 11.4, and it follows from Proposition 11.2(e) that

$$M \mid M{\downarrow}_H^G{\uparrow}_H^G .$$

Hence

$$M \mid M{\downarrow}_H^G{\uparrow}_H^G \mid (KH)^n{\uparrow}_H^G \cong (KG)^n ,$$

so $M$ is projective. ■

## 12 Vertices and Sources

As in the case in which $KG$ is semisimple, relative projectivity is just projectivity, we now focus on the non-semiminple case.

> For the remainder of this chapter, we assume that $\text{char}(K) =: p > 0$ and $p \mid |G|$.

As said before, we now want to explain some techniques that are available to understand indecomposable modules better. Vertices and sources are two parameters making this possible.

**Theorem 12.1**

Let $M$ be an indecomposable $KG$-module.

(a) There is a unique conjugacy class of subgroups $Q$ of $G$ which are minimal subject to the property that $M$ is $Q$-projective.

(b) Let $Q$ be a minimal subgroup of $G$ such that $M$ is $Q$-projective. Then, there exists an indecomposable $KQ$-module $T$ which is unique, up to conjugacy by elements of $N_G(Q)$, such that $M$ is a direct summand of $T{\uparrow}_Q^G$. Such a $KQ$-module $T$ is necessarily a direct summand of $M{\downarrow}_Q^G$.

This characterisation leads us to the following definition:

**Definition 12.2**

Let $M$ be an indecomposable $KG$-module.

(a) A **vertex** of $M$ is a minimal subgroup $Q$ of $G$ such that $M$ is relatively $Q$-projective. The set of all vertices of $M$ is denoted by $\mathrm{vtx}(M)$.

(b) Given a vertex $Q$ of $M$, a **$KQ$-source**, or simply a **source** of $M$ is a $KQ$-module $T$ such that $M \mid T{\uparrow}_Q^G$.

**Remark 12.3**

(a) A vertex $Q$ of an indecomposable $KG$-module $M$ is not uniquely defined, in general. However, the vertices of $M$ are unique **up to $G$-conjugacy**, so in particular are all isomorphic. For this reason, in general, one (i.e. you!) should **never** talk about *the* vertex of a module (of course, unless a vertex has been fixed). We either say that $Q$ *is a vertex of* $M$, or talk about *the vertices of* $M$. (Unfortunately many textbooks/articles are very sloppy with this terminology, inducing errors.)

(b) For a fixed vertex $Q$ of $M$, a source of $M$ is defined up to conjugacy by elements of $N_G(Q)$.

**Warning!** Vertices and sources are very useful theoretical tools in general, but extremely difficult to compute concretely. However, the following properties are useful.

To begin with, by Theorem 11.4, we know that every $KG$-module is projective relative to a Sylow $p$-subgroup of $G$. Therefore, by minimality, vertices are contained in Sylow $p$-subgroups. Hence:

**Proposition 12.4**

The vertices of an indecomposable $KG$-module are $p$-subgroups of $G$.

**Proposition 12.5**

Let $U$ be an indecomposable $KG$-module and let $Q \in \mathrm{vtx}(U)$. If $P \in \mathrm{Syl}_p(G)$ is such that $Q \subseteq P$, then

$$|P : Q| \mid \dim_K(U)\,.$$

In particular if $U$ is a PIM of $KG$, then $|P| = |G|_p \mid \dim_K(U)$.

**Example 8**

(a) The trivial subgroup $\{1\}$ is a vertex of an indecomposable $KG$-module $U \iff U$ is a PIM of $KG$.

(b) The vertices of the trivial $KG$-module are the Sylow $p$-subgroups of $G$, i.e. $\mathrm{vtx}(K) = \mathrm{Syl}_p(G)$, and all sources are trivial.

**Exercise 12.6**

Prove that the vertices of any $KG$-module with $K$-dimension coprime to $p$ are the Sylow $p$-subgroups of $G$.

Conceptually, the closer the vertices of a module are to the trivial subgroup, the closer this module is to being projective.

Finally, we give a name to the modules which have a trivial source. We will see in Lecture 4 that these module play a particularly important role in block theory.

**Definition 12.7 (*trivial source module*)**

A $KG$-module is called a **trivial source** $KG$-module if it is a finite direct sum of $KG$-modules with a trivial source $K$.

**Warning!** Some texts (books/articles/...) require that a trivial source module is indecomposable, others do not.

## 13 The Green Correspondence

The Green correspondence is a correspondence which relates the indecomposable $KG$-modules with a fixed vertex with the indecomposable $KL$-modules with the same vertex for well-chosen subgroups $L \leqslant G$. It is used to reduce questions about indecomposable modules to a situation where a vertex of the given indecomposable module is a normal subgroup.

**Theorem 13.1 (*Green Correspondence*)**

Let $Q$ be a $p$-subgroup of $G$ and let $L$ be a subgroup of $G$ containing $N_G(Q)$.

(a) If $U$ is an indecomposable $KG$-module with vertex $Q$, then

$$U\downarrow_L^G = f(U) \oplus X$$

where $f(U)$ is the unique indecomposable direct summand of $U\downarrow_L^G$ with vertex $Q$ and every direct summand of $X$ is $L \cap {}^xQ$-projective for some $x \in G \backslash L$.

(b) If $V$ is an indecomposable $KL$-module with vertex $Q$, then

$$V\uparrow_L^G = g(V) \oplus Y$$

where $g(V)$ is unique indecomposable direct summand of $V\uparrow_L^G$ with vertex $Q$ and every direct summand of $Y$ is $Q \cap {}^xQ$-projective for some $x \in G \backslash L$.

(c) With the notation of (a) and (b), we then have $g(f(U)) \cong U$ and $f(g(V)) \cong V$. In other words, $f$ and $g$ define a bijection

$$\begin{array}{ccc} \left\{\begin{matrix}\text{isomorphism classes of indecomposable} \\ KG\text{-modules with vertex } Q\end{matrix}\right\} & \stackrel{\sim}{\longleftrightarrow} & \left\{\begin{matrix}\text{isomorphism classes of indecomposable} \\ KL\text{-modules with vertex } Q\end{matrix}\right\} \\ U & \mapsto & f(U) \\ g(V) & \leftarrow\!\shortmid & V\,. \end{array}$$

Moreover, corresponding modules have a source in common.

**Terminology**: $f(U)$ is called the $KL$-**Green correspondent** of $U$ (or simply the **Green correspondent**) and $g(V)$ is called the $KG$-**Green correspondent of** $V$ (or simply the **Green correspondent** of $V$).

**Warning!** When working with the Green correspondence it is essential that a vertex $Q$ is fixed and not considered up to conjugation, because the $G$-conjugacy class of $Q$ and the $L$-conjugacy class of $Q$ do not coincide in general.

**Example 9**

The Green correspondent of the trivial module is the trivial module, for $K\downarrow_L^G = K$.

## 14 $p$-Permutation Modules

**Definition 14.1 (*Permutation module*)**

A $KG$-module is called a **permutation** $KG$-module if it admits a $K$-basis $X$ which is invariant under the action of the group $G$. We denote this module by $KX$.

Permutation $KG$-modules and, in particular, their indecomposable direct summands have remarkable properties, which we investigate in this section.

**Remark 14.2**

If $KX$ is a permutation $KG$-module on $X$, then a decomposition of the basis $X$ as a disjoint union of $G$-orbits, say $X = \bigsqcup_{i=1}^n X_i$, yields a direct sum decomposition of $KX$ as a $KG$-module as

$$KX = \bigoplus_{i=1}^{n} KX_i\,.$$

Thus, we can assume that $X$ is a transitive $G$-set, in which case we have a direct sum decomposition as a $K$-vector space

$$KX = \bigoplus_{g\in[G/H]} Kgx$$

where $H := \mathrm{Stab}_G(x)$, the stabiliser in $G$ of some $x \in X$, and $G$ acts transitively on the summands. Hence,

$$KX \cong K\uparrow_H^G\,.$$

It follows that an arbitrary permutation $KG$-module is isomorphic to a direct sum of $KG$-modules of the form $K\uparrow_H^G$ for various $H \leqslant G$.

Conversely, an induced module of the form $K\uparrow_H^G$ ($H \leqslant G$) is always a permutation $KG$-module. Indeed, as $K\uparrow_H^G = KG \otimes_{KH} K = \bigoplus_{g\in[G/H]} g \otimes K$ as $K$-vector space, it has on obvious $G$-invariant $K$-basis given by the set

$$\{g \otimes 1_K \mid g \in [G/H]\}\,.$$

In fact, more generally if $H \leqslant G$ and $KX$ is a permutation $KH$-module on $X$, then $KX\uparrow_H^G$ is a permutation $KG$-module with $G$-invariant $K$-basis $\{g \otimes x \mid g \in [G/H], x \in X\}$. In other words, induction preserves permutation modules.

**Exercise 14.3**

Prove that direct sums, restriction, inflation and conjugation also preserve permutation modules.

In order to understand the indecomposable direct summands of the permutation $KG$-modules, we observe that they all have a trivial source and we will apply the Green correspondence to see that, up to isomorphism, there are only a finite number of them.

**Proposition-Definition 14.4 (*p-permutation module*)**

Let $M$ be a $KG$-module and let $P \in \mathrm{Syl}_p(G)$. Then, the following conditions are equivalent:

(a) $M{\downarrow}_Q^G$ is a permutation $KQ$-module for each $p$-subgroup $Q \leqslant G$;

(b) $M{\downarrow}_P^G$ is a permutation $KP$-module;

(c) $M$ has a $K$-basis which is invariant under the action of $P$;

(d) $M$ is isomorphic to a direct summand of a permutation $KG$-module;

(e) $M$ is a trivial source $KG$-module.

If $M$ fulfils one of these equivalent conditions, then it is called a ***p*-permutation** $KG$-module.

**Note.** In fact *p-permutation* $KG$-modules and *trivial source* $KG$-modules are two different pieces of terminology for the same concept. French/German speaking authors tend to favour the terminology *p-permutation module* (and reserve the terminology *trivial source module* for an indecomposable module with a trivial source), whereas English speaking authors tend to favour the terminology *trivial source module*.

**Exercise 14.5**

Prove that $p$-permutation modules are preserved by the following operations: direct sums, tensor products, restriction, inflation, conjugation, induction.

**Example 10**

(a) If $G$ is a $p$-group, then any $p$-permutation module is a permutation module.

(b) The PIMs of $KG$ are precisely the $KG$-modules with vertex $\{1\}$ and trivial source, so any projective $KG$-module is a $p$-permutation $KG$-module.

**Example 11**

Any $KG$-modules $Y$ of $K$-dimension 1 is a $p$-permutation module.
Proof. Let $Q$ be a vertex of $Y$ and let $f(Y)$ be the $kN_G(Q)$-Green correspondent of $Y$. Then clearly $\dim_K f(Y) = 1$ as well. Thus $f(Y)$ is a simple and therefore has a trivial $KQ$-source. Indeed, $f(Y){\downarrow}_Q^{N_G(Q)}$ is semisimple by the weak version of Clifford's Theorem, and so must be a direct sum of copies of the trivial $kQ$-module because $Q$ is a $p$-group and therefore has only one simple module, up to isomorphism, namely the trivial module.

Generalising these examples, we can characterise the indecomposable $p$-permutation $KG$-modules with a given vertex $Q \leqslant G$ as described below.

**Example 12 (*Green Correspondence applied to indecomposables with a trivial source*)**

(1) If $M$ is an indecomposable $p$-permutation $KG$-module with vertex $Q \leqslant G$, then $Q$ acts trivially on the $KN_G(Q)$-Green correspondent $f(M)$ of $M$. Thus $f(M)$ can be viewed as a $K[N_G(Q)/Q]$-module. As such, $f(M)$ is indecomposable and projective.

(2) Conversely, if $N$ is a projective indecomposable $K[N_G(Q)/Q]$-module, then $\mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(N)$ is an indecomposable $KN_G(Q)$-module with vertex $Q$ and trivial source. Its $KG$-Green correspondent is then also an indecomposable $KG$-module with vertex $Q$ and trivial source, hence is an indecomposable $p$-permutation $KG$-module

(3) In this way we obtain a bijection

$$\left\{\begin{matrix}\text{isomorphism classes of indecomposable}\\ p\text{-permutation } KG\text{-modules with vertex } Q\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{isomorphism classes of projective}\\ \text{indecomposable } K[N_G(Q)/Q]\text{-modules}\end{matrix}\right\}.$$