A Short Introduction to the Modular Representation Theory of Finite Groups

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Contents

Foreword

This text is a transcript of my lectures for the mini-course **Introduction to the Modular Representation Theory of Finite Groups** held at the **IKIU Summer School on Representation Theory 2022** and which took place from the 5th of September to the 7th of September 2022 in Qazin, Iran (3 x 80 minutes).

Together with the necessary theoretical foundations the main aim of this mini-course was to provide the participants with an introduction to the representation theory of finite groups from a module- and block-theoretic point of view.

The material presented here has been very much influenced by lectures and summer lectures available in conference proceedings or which I have followed myself as a student, doctoral student or young postdoc. In particular, I want to mention here:

- [Mal15] G. Malle, Darstellungstheorie, M.Sc. lecture, TU Kaiserslautern, WS 2015/16. [Unpublished]
- [Bro92] M. Broué, Equivalences of blocks of group algebras. Ottawa, 1992. `[MathSciNet]`
- [Kes07] R. Kessar, Introducton to block theory. EPFL: CIB 2005 & YAC 2012. `[MathSciNet]`
- [Kue18] B. Külshammer, Basic local representation theory. EPFL/CIB, 2016. `[MathSciNet]`
- [HKK10] G. Hiss, R. Kessar, B. Külshammer, An Introduction to the Representation Theory of Finite Groups. Aachen, 2010. [Unpublished]
- [Samb16] B. Sambale, Determination of block invariants. EPFL/CIB, 2016. [Unpublished, available from his webpage]

Acknowledgement:
First of all, I wish to thank the organisers of the *IKIU Summer School on Representation Theory* for inviting me to hold this mini-course and the efforts they invested in organising this event as a *presence* event. I believe, after 2 years of restrictions due to the pandemics, all participants immensely appreciated this opportunity to meet their contemporaries and hear about advances in current research problems.

Conventions

Unless otherwise stated, throughout these notes we make the following general assumptions:

- all groups considered are **finite**;
- all rings considered are **associative** and **unital**, i.e. possess a neutral element for the multiplication, denoted by 1;
- all modules considered are finitely generated **left** modules;

- R always denotes an associative ring with a 1;
- G always denotes a finite group;
- K always denotes a field of arbitrary characteristic;
- A always denotes a finite-dimensional K-algebra, which is split.
- $(F, \mathcal{O}, k)$ always denotes a splitting p-modular system for G and its subgroups, where F contains a primitive $\exp(G)$-th root of 1.

Representation theory of finite groups is originally concerned with the ways of writing a finite group G as a group of matrices, that is using group homomorphisms from G to the general linear group $\mathrm{GL}_n(K)$ of invertible $n \times n$-matrices with coefficients in a field K for some positive integer n. Thus, we shall first define representations of groups using this approach, and then translate such homomorphisms $G \longrightarrow \mathrm{GL}_n(K)$ into the language of module theory.

Notation. Throughout this chapter, unless otherwise specified, $\mathrm{char}(K) \geqslant 0$. Moreover, we assume that all KG-modules considered are finitely generated, so finite-dimensional when regarded as K-vector spaces.

1 Linear Representations of Finite Groups

To begin with, we review elementary definitions and examples about representations of finite groups.

Definition 1.1 (*K-representation, matrix representation*)

(a) A **K-representation** of G is a group homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$, where $V \cong K^n$ ($n \in \mathbb{Z}_{\geqslant 0}$) is a K-vector and $\mathrm{GL}(V) := \mathrm{Aut}_K(V)$.

(b) A **matrix representation** of G over K is a group homomorphism $X : G \longrightarrow \mathrm{GL}_n(K)$ ($n \in \mathbb{Z}_{\geqslant 0}$).

In both cases the integer n is called the **degree** of the representation.

(c) A K-representation (resp. a matrix representation) is called an **ordinary representation** if $\mathrm{char}(K) = 0$ (or more generally if $\mathrm{char}(K) \nmid |G|$), and it is called a **modular representation** if $\mathrm{char}(K) \mid |G|$.

Both concepts of a representation and of a matrix representation are closely related. Indeed, choosing a K-basis B of V, then we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, dashed, "\exists"'] & \mathrm{GL}(V) \arrow[d, leftrightarrow, "\cong"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{GL}_n(K)\,. \end{tikzcd}$$

Example 1

(a) The map

$$\begin{array}{rccl} \rho : & G & \longrightarrow & \mathrm{GL}(K) \cong K^\times \\ & g & \mapsto & \mathrm{Id}_K \leftrightarrow 1_K \end{array}$$

is a K-representation of G of degree 1, called **the trivial representation** of G.

(b) If X is a finite G-set, i.e. a finite set endowed with a left action $\cdot : G \times X \longrightarrow X$, and V is a K-vector space with basis $\{e_x \mid x \in X\}$, then

$$\begin{array}{rccl} \rho_X : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_X(g) : V \longrightarrow V, e_x \mapsto e_{g \cdot x} \end{array}$$

is a K-representation of G of degree $|X|$, called the **permutation representation** associated with X.

Two particularly interesting examples are the following:

(1) if $G = S_n$ ($n \geqslant 1$) is the symmetric group on n letters, $X = \{1, 2, \dots, n\}$, and the left action $\cdot : G \times X \longrightarrow X$ is given by the natural action of S_n then ρ_X is called **natural representation** of S_n;

(2) if $X = G$ and the left action $\cdot : G \times X \longrightarrow X$ is just the multiplication in G, then $\rho_X =: \rho_{\mathrm{reg}}$ is called the **regular representation** of G.

Definition 1.2 (*Homomorphism of representations, equivalent representations*)

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two K-representations of G.

(a) A K-homomorphism $\alpha : V_1 \longrightarrow V_2$ such that $\rho_2(g) \circ \alpha = \alpha \circ \rho_1(g)$ for each $g \in G$ is called a **homomorphism of representations** (or a G**-homomorphism**) between ρ_1 and ρ_2.

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & V_1 \arrow[d, "\alpha"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

(b) If, moreover, α is a K-isomorphism, then it is called an **isomorphism of representations** (or a G**-isomorphism**), and the K-representations ρ_1 and ρ_2 are called **equivalent** (or **similar**, or **isomorphic**). In this case we write $\rho_1 \sim \rho_2$.

Remark 1.3

(a) Equivalent representations have the same degree.

(b) Clearly $\sim$ is an equivalence relation.

(c) In consequence, it essentially suffices to study representations up to equivalence (as it essentially suffices to study groups up to isomorphism).

Definition 1.4 (*G-invariant subspace, irreducibility, subrepresentation*)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation of G.

(a) A K-subspace $W \subseteq V$ is called **G-invariant** if

$$\rho(g)\big(W\big) \subseteq W \qquad \forall g \in G\,.$$

(In fact in this case the reverse inclusion holds as well, since for each $w \in W$ we can write $w = \rho(gg^{-1})(w) = \rho(g)\big(\rho(g^{-1})(w)\big) \in \rho(g)\big(W\big)$, hence $\rho(g)\big(W\big) = W$.)

(b) The representation ρ is called **irreducible** if it admits exactly two G-invariant K-subspaces, namely 0 and V itself; it is called **reducible** if it is not irreducible.

(c) If $0 \subsetneq W \subseteq V$ is a G-invariant K-subspace, then

$$\begin{array}{rccl} \rho_W : & G & \longrightarrow & \mathrm{GL}(W) \\ & g & \mapsto & \rho_W(g) := \rho(g)|_W : W \longrightarrow W \end{array}$$

is called a **subrepresentation** of ρ. (This is clearly again a K-representation of G.)

2 The Group Algebra and its Modules

We actually want to be able to see K-representations of a group G as *modules*.

Definition 2.1 (*Group algebra*)

The **group algebra of G over K** is the K-algebra KG whose elements are the K-linear combinations $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in K\ \forall g \in G$, and addition and multiplication are given by

$$\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g \quad \text{ and } \quad \big(\sum_{g\in G} \lambda_g g\big) \cdot \big(\sum_{h\in G} \mu_h h\big) = \sum_{g,h\in G} (\lambda_g \mu_h) gh$$

respectively.

Remark 2.2

- $1_{KG} = 1_G$;
- $\dim_K(KG) = |G|$;
- KG is commutative if and only if G is an abelian group;
- as K is a field, KG is a left Artinian ring, so by Hopkins' Theorem a KG-module is finitely generated if and only if it admits a composition series.

Also notice that since G is a group, the map $KG \longrightarrow KG$ defined by $g \mapsto g^{-1}$ for each $g \in G$ is an anti-automorphism. It follows that any *left* KG-module M may be regarded as a *right* KG-module via the right G-action $m \cdot g := g^{-1} \cdot m$. Thus the sidedness of KG-modules is not usually an issue.

As KG is a K-algebra, we may of course consider modules over KG and we recall that any KG-module is in particular a K-vector space. Moreover, we adopt the following convention, which is automatically satisfied if K is a field.

Proposition 2.3

(a) Any K-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ of G gives rise to a KG-module structure on V, where the external composition law is defined by the map

$$\begin{array}{rccl} \cdot : & G \times V & \longrightarrow & V \\ & (g, v) & \mapsto & g \cdot v := \rho(g)(v) \end{array}$$

extended by K-linearity to the whole of KG.

(b) Conversely, every KG-module $(V, +, \cdot)$ defines a K-representation

$$\begin{array}{rccl} \rho_V : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_V(g) : V \longrightarrow V, v \mapsto \rho_V(g) := g \cdot v \end{array}$$

of the group G.

Example 2

Via Proposition 2.3 the trivial representation (Example 1(a)) corresponds to the so-called **trivial KG-module**, that is, K itself seen as a KG-module via the G-action

$$\cdot : G \times K \longrightarrow K$$
$$(g, \lambda) \longmapsto g \cdot \lambda := \lambda$$

extended by K-linearity to the whole of KG .

Exercise 2.4

Prove that the regular representation ρ_{reg} of G defined in Exampale 1(b)(2) corresponds to the regular KG-module KG° via Proposition 2.3.

Convention: In the sequel, when no confusion is to be made, we drop the $\circ$-notation to denote the regular KG-module and simply write KG instead of KG°.

Remark 2.5 (*Dictionary*)

More generally, through Proposition 2.3, we may transport terminology and properties from KG-modules to K-representations and conversely.

This lets us build the following translation **dictionary**:

K-REPRESENTATIONS		KG-MODULES
K-representation of G	$\longleftrightarrow$	KG-module
degree	$\longleftrightarrow$	K-dimension
homomorphism of K-representations	$\longleftrightarrow$	homomorphism of KG-modules
equivalent K-representations	$\longleftrightarrow$	isomorphism of KG-modules
subrepresentation	$\longleftrightarrow$	KG-submodule
direct sum of representations $\rho_{V_1} \oplus \rho_{V_2}$	$\longleftrightarrow$	direct sum of KG-modules $V_1 \oplus V_2$
irreducible representation	$\longleftrightarrow$	simple (= irreducible) KG-module
the trivial representation	$\longleftrightarrow$	the trivial KG-module K
the regular representation of G	$\longleftrightarrow$	the regular KG-module KG°
completely reducible K-representation	$\longleftrightarrow$	semisimple KG-module (= completely reducible)
every K-representation of G is completely reducible	$\longleftrightarrow$	KG is semisimple
...		...

3 Operations on Groups and Modules

Next we see how to construct new KG-modules from old ones using standard module operations such as tensor products, Hom-functors, duality, or using subgroups or quotients of the initial group.

Remark 3.1 (*Tensors,* Hom*'s and duality*)

Let M and N be KG-modules.

(a) The tensor product $M \otimes_K N$ of M and N balanced over K becomes a KG-module via the **diagonal action** of G. In other words, the external composition law is defined by the G-action

$$\begin{array}{rrcl} \cdot: & G \times (M \otimes_K N) & \longrightarrow & M \otimes_K N \\ & (g, m \otimes n) & \mapsto & g \cdot (m \otimes n) := gm \otimes gn \end{array}$$

extended by K-linearity to the whole of KG.

(b) The abelian group $\mathrm{Hom}_K(M, N)$ becomes a KG-module via the so-called **conjugation action** of G. In other words, the external composition law is defined by the G-action

$$\begin{array}{rrcl} \cdot: & G \times \mathrm{Hom}_K(M, N) & \longrightarrow & \mathrm{Hom}_K(M, N) \\ & (g, f) & \mapsto & g \cdot f : M \longrightarrow N, m \mapsto (g \cdot f)(m) := g \cdot f(g^{-1} \cdot m) \end{array}$$

extended by K-linearity to the whole of KG.

(c) Specifying Definition 3.1 to $N = K$ yields a KG-module structure on the K-dual $M^* = \mathrm{Hom}_K(M, K)$, that is, M^* becomes a KG-module via the external composition law is defined by the map

$$\begin{array}{rccl} \cdot: & G \times M^* & \longrightarrow & M^* \\ & (g,f) & \mapsto & g \cdot f : M \longrightarrow K, m \mapsto (g \cdot f)(m) := f(g^{-1} \cdot m) \end{array}$$

extended by K-linearity to the whole of KG.

On the other hand, we may define new module structures from known ones for subgroups, overgroups and quotients. This leads to standard operations called *restriction*, *inflation*, and *induction*.

Remark 3.2

(a) If $H \leqslant G$ is a subgroup, then the inclusion $H \longrightarrow G, h \mapsto h$ can be extended by K-linearity to an injective algebra homomorphism $\iota : KH \longrightarrow KG, \sum_{h\in H} \lambda_h h \mapsto \sum_{h\in H} \lambda_h h$. Hence KH is a K-subalgebra of KG.

(b) Similarly, if $U \trianglelefteq G$ is a normal subgroup, then the quotient homomorphism $G \longrightarrow G/U$, $g \mapsto gU$ can be extended by K-linearity to an algebra homomorphism $\pi : KG \longrightarrow K[G/U]$.

It is clear that we can always perform changes of the base ring using the above homomorphism in order to obtain new module structures. This yields:

Definition 3.3 (*Restriction*)

Let $H \leqslant G$ be a subgroup. If M is a KG-module, then M may be regarded as a KH-module through a change of the base ring along $\iota : KH \longrightarrow KG$, which we denote by $\mathrm{Res}^G_H(M)$ or simply $M\!\downarrow^G_H$ and call the **restriction** of M from G to H.

Definition 3.4 (*Inflation*)

Let $U \trianglelefteq G$ be a normal subgroup. If M is a $K[G/U]$-module, then M may be regarded as a KG-module through a change of the base ring along $\pi : KG \longrightarrow K[G/U]$, which we denote by $\mathrm{Inf}^G_{G/U}(M)$ and call the **inflation** of M from G/U to G.

Lemma 3.5

(a) If $H \leqslant G$ and M_1, M_2 are two KG-modules, then $(M_1 \oplus M_2)\downarrow^G_H = M_1\downarrow^G_H \oplus M_2\downarrow^G_H$. If $U \trianglelefteq G$ and M_1, M_2 are two $K[G/U]$-modules, then $\mathrm{Inf}^G_{G/U}(M_1 \oplus M_2) = \mathrm{Inf}^G_{G/U}(M_1) \oplus \mathrm{Inf}^G_{G/U}(M_2)$.

(b) **(Transitivity of restriction)** If $L \leqslant H \leqslant G$ and M is a KG-module, then $M\downarrow^G_H\downarrow^H_L = M\downarrow^G_L$.

(c) If $H \leqslant G$ and M is a KG-module, then $(M^*)\downarrow^G_H \cong (M\downarrow^G_H)^*$. If $U \trianglelefteq G$ and M is a $K[G/U]$-module, then $\mathrm{Inf}^G_{G/U}(M^*) \cong (\mathrm{Inf}^G_{G/U} M)^*$.

A third natural operation comes from extending scalars from a subgroup to the initial group.

Definition 3.6 (*Induction*)

Let $H \leqslant G$ be a subgroup and let M be a KH-module. Regarding KG as a (KG, KH)-bimodule, we define the **induction** of M from H to G to be the left KG-module

$$M\uparrow^G_H = \mathrm{Ind}^G_H(M) := KG \otimes_{KH} M\,.$$

Example 3

(a) If $H = \{1\}$ and $M = K$, then $K\uparrow_{\{1\}}^{G} = KG \otimes_K K \cong KG$.

(b) **Transitivity of induction**: clearly $L \leqslant H \leqslant G$ and M is a KL-module, then $M\uparrow_L^G = (M\uparrow_L^H)\uparrow_H^G$ by the associativity of the tensor product.

First, we analyse the structure of an induced module in terms of the left cosets of H.

Remark 3.7

Writing $[G/H]$ $(= \{g_1, \dots, g_{|G:H|})$ for a set of representatives of the left cosets, given a KH-module M, we have

$$KG \otimes_{KH} M = (\bigoplus_{g\in[G/H]} gKH) \otimes_{KH} M = \bigoplus_{g\in[G/H]} (gKH \otimes_{KH} M) = \bigoplus_{g\in[G/H]} (g \otimes M)\,,$$

where we set

$$g \otimes M := \{g \otimes m \mid m \in M\} \subseteq KG \otimes_{KH} M\,.$$

Clearly, each $g \otimes M \cong M$ as a K-space via the K-isomorphism $g \otimes M \longrightarrow M, g \otimes m \mapsto m$, so

$$\dim_K(\mathrm{Ind}_H^G(M)) = |G : H| \cdot \dim_K(M)\,.$$

Theorem 3.8 (*Adjunction / Frobenius reciprocity / Nakayama relations*)

Let $H \leqslant G$. Let N be a KG-module and let M be a KH-module. Then, there are K-isomorphisms:

(a) $\mathrm{Hom}_{KH}(M, \mathrm{Hom}_{KG}(KG, N)) \cong \mathrm{Hom}_{KG}(KG \otimes_{KH} M, N)$,
or in other words, $\mathrm{Hom}_{KH}(M, N\downarrow_H^G) \cong \mathrm{Hom}_{KG}(M\uparrow_H^G, N)$;

(b) $\mathrm{Hom}_{KH}(N\downarrow_H^G, M) \cong \mathrm{Hom}_{KG}(N, M\uparrow_H^G)$.

Proposition 3.9

Let $H \leqslant G$ be a subgroup. Let N be a KG-module and let M be a KH-module. Then, there are KG-isomorphisms:

(a) $(M \otimes_K N\downarrow_H^G)\uparrow_H^G \cong M\uparrow_H^G \otimes_K N$; and

(b) $\mathrm{Hom}_K(M, N\downarrow_H^G)\uparrow_H^G \cong \mathrm{Hom}_K(M\uparrow_H^G, N)$.

Finally, if H and L are subgroups of G, we wish to describe what happens if we induce a KL-module from L to G and then restrict it to H.

Now if M is a KL-module, we will also write ${}^gM$ for $g \otimes M$, which is a left $K({}^gL)$-module with

$$(glg^{-1}) \cdot (g \otimes m) = g \otimes lm$$

for each $l \in L$ and each $m \in M$.

Theorem 3.10 (*Mackey formula*)

Let $H, L \leqslant G$ and let M be a KL-module. Then, as KH-modules,

$$M\uparrow_L^G\downarrow_H^G \cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\downarrow^{{}^gL}_{H\cap {}^gL})\uparrow^H_{H\cap {}^gL} \ .$$

Proof: We need to examine KG seen as a (KH, KL)-bimodule (i.e. with left and right external laws given by multiplication in G). Since $G = \bigsqcup_{g\in[H\backslash G/L]} HgL$, we have

$$KG = \bigoplus_{g\in[H\backslash G/L]} K\langle HgL\rangle$$

as (KH, KL)-bimodule, where $K\langle HgL\rangle$ denotes the K-vector space with K-basis the double coset HgL. Also

$$K\langle HgL\rangle \cong KH \otimes_{K(H\cap {}^gL)} (g \otimes KL)\,,$$

where $hgl \in HgL$ corresponds to $h \otimes g \otimes l$. It follows that as left KH-modules we have

$$\begin{aligned} M\uparrow_L^G\downarrow_H^G \cong (KG \otimes_{KL} M)\downarrow_H^G &\cong \bigoplus_{g\in[H\backslash G/L]} K\langle HgL\rangle \otimes_{KL} M \\ &\cong \bigoplus_{g\in[H\backslash G/L]} KH \otimes_{K(H\cap {}^gL)} (g\otimes KL) \otimes_{KL} M \\ &\cong \bigoplus_{g\in[H\backslash G/L]} KH \otimes_{K(H\cap {}^gL)} (g\otimes M)\downarrow^{{}^gL}_{H\cap {}^gL} \\ &\cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\downarrow^{{}^gL}_{H\cap {}^gL})\uparrow^H_{H\cap {}^gL} \ . \end{aligned}$$

■

Monday. Chapter 2. Semisimplicity and Simplicity

The aim of this chapter is to study two important classes of modules over the group algebra, namely *simple modules* and *semisimple modules*. In particular, our first aim is to understand what the general theory of semisimple rings and the Artin-Wedderburn theorem bring to the theory of representations of finite groups over a field of arbitrary characteristic.

Notation. From now on, we let $\mathrm{Irr}(R) := \{\text{isomorphism classes of simple } R\text{-modules}\}$.

4 Schur's Lemma

Schur's Lemma is one of the most basic result, which lets us understand homomorphisms between *simple* modules, and, more importantly, endomorphisms of such modules. It is

Theorem 4.1 (*Schur's Lemma*)

(a) Let V, W be simple R-modules. Then:

  (i) $\mathrm{End}_R(V)$ is a skew-field, and

  (ii) if $V \not\cong W$, then $\mathrm{Hom}_R(V, W) = 0$.

(b) If K is an algebraically closed field, A is a K-algebra, and V is a simple A-module, then

$$\mathrm{End}_A(V) = \{\lambda\,\mathrm{Id}_V \mid \lambda \in K\} \cong K\,.$$

Remark 4.2

In (b) the assumption that the field K is algebraically closed is in general too strong and we often replace this hypothesis by the hypothesis that the algebra A is **split**, meaning that

$$\mathrm{End}_A(S) \cong K \quad \text{for every simple } A\text{-module } S\,.$$

In this respect, the field K is a **splitting field** for G if the group algebra KG is split. This will be one of our standard assumptions.

From now on, we assume that K is a splitting field for G.

5 The Artin-Wedderburn Structure Theorem

The next step is to analyse semisimple rings and modules, sorting simple modules into isomorphism classes and relate these to a direct summand of the regular module.

Definition 5.1

If M is a semisimple R-module and S is a simple R-module, then the S-**homogeneous component** of M, denoted $S(M)$, is the sum of all simple R-submodules of M isomorphic to S.

Theorem 5.2 (*Wedderburn*)

If R is a semisimple ring, then the following assertions hold.

(a) If $S \in \mathrm{Irr}(R)$, then $S(R^\circ) \neq 0$. Furthermore, $|\,\mathrm{Irr}(R)| < \infty$.

(b) We have
$$R^\circ = \bigoplus_{S \in \mathrm{Irr}(R)} S(R^\circ)\,,$$
where each homogenous component $S(R^\circ)$ is a two-sided ideal of R and $S(R^\circ)T(R^\circ) = 0$ if $S \neq T \in \mathrm{Irr}(R)$.

(c) Each $S(R^\circ)$ is a simple left Artinian ring, the identity element of which is an idempotent element of R lying in $Z(R)$.

Remark 5.3

Remember that if R is a semisimple ring, then the regular module R° admits a composition series. Therefore it follows from the Jordan-Hölder Theorem that
$$R^\circ = \bigoplus_{S \in \mathrm{Irr}(R)} S(R^\circ) \cong \bigoplus_{S \in \mathrm{Irr}(R)} \bigoplus_{i=1}^{n_S} S$$
for uniquely determined integers $n_S \in \mathbb{Z}_{>0}$.

Theorem 5.4 (*Artin-Wedderburn*)

If R is a semisimple ring, then, as a ring,
$$R \cong \prod_{S \in \mathrm{Irr}(R)} M_{n_S}(D_S)\,,$$
where $D_S := \mathrm{End}_R(S)^{\mathrm{op}}$ is a division ring.

> Let us now assume that $R = A$ is a split K-algebra.

We obtain the following Corollary to Wedderburn's and Artin-Wedderburn's Theorems.

Theorem 5.5

Assume A is semisimple and let $S \in \mathrm{Irr}(A)$ be a simple A-module. Then the following statements hold:

(a) $S(A^\circ) \cong M_{n_S}(K)$ and $\dim_K(S(A^\circ)) = n_S^2$;

(b) $\dim_K(S) = n_S$;

(c) $\dim_K(A) = \sum_{S\in\mathrm{Irr}(A)} \dim_K(S)^2$;

(d) $|\,\mathrm{Irr}(A)| = \dim_K(Z(A))$.

Exercise 5.6

Prove Thm. 5.5.

Corollary 5.7

Up to isomorphism, the number of simple A-modules is $|\,\mathrm{Irr}(A)| = \dim_K(Z(A/J(A)))$.

Proof: Since A and $A/J(A)$ have the same simple modules $|\,\mathrm{Irr}(A)| = |\,\mathrm{Irr}(A/J(A))|$. Moreover, the quotient $A/J(A)$ is J-semisimple, hence semisimple because finite-dimensional algebras are left Artinian rings. Therefore it follows from Theorem 5.5(d) that

$$|\,\mathrm{Irr}(A)| = |\,\mathrm{Irr}(A/J(A))| = \dim_K\left(Z(A/J(A))\right).$$

■

Corollary 5.8

If A is commutative, then any simple A-module has K-dimension 1.

Proof: First assume that A is semisimple. As A is commutative, $A = Z(A)$. Hence parts (d) and (c) of Theorem 5.5 yield

$$|\,\mathrm{Irr}(A)| = \dim_K(A) = \sum_{S\in\mathrm{Irr}(A)} \underbrace{\dim_K(S)^2}_{\geqslant 1},$$

which forces $\dim_K(S) = 1$ for each $S \in \mathrm{Irr}(A)$.
Now, if A is not semissimple, then again we use the fact that A and $A/J(A)$ have the same simple modules. Because $A/J(A)$ is semisimple and also commutative, the argument above tells us that all simple $A/J(A)$-modules have K-dimension 1. The claim follows. ■

Applying these results to the group algebra KG, we obtain for example that:

Corollary 5.9

There are only finitely many isomorphism classes of simple KG-modules.

Proof: The claim follows directly from Corollary 5.7. ■

Corollary 5.10

If G is an abelian group then any simple KG-module is one-dimensional.

Proof: Since KG is commutative the claim follows directly from Corollary 5.8. ■

Corollary 5.11

Let p be a prime number. If G is a p-group, and $\mathrm{char}(K) = p$, then the trivial module is the unique simple KG-module, up to isomorphism.

Proof: Because G is a p-group, we have $J(KG) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\} =: I(KG)$ (the augmentation ideal (see definition in Exercise 6.3), so $KG/J(KG) \cong K$ as K-algebras. Now, as K is commutative, $Z(K) = K$, and it follows from Corollary 5.7 that

$$|\,\mathrm{Irr}(KG)| = \dim_K Z(KG/J(KG)) = \dim_K K = 1\,.$$

■

6 Semisimplicity of the Group Algebra

The semisimplicity of the group algebra depends on both the characteristic of the field and the order of the group. This is Maschke's Theorem and its converse.

Theorem 6.1 (*Maschke*)

If $\mathrm{char}(K) \nmid |G|$, then KG is a semisimple K-algebra.

Example 4

If $K = \mathbb{C}$ is the field of complex numbers, then $\mathbb{C}G$ is a semisimple $\mathbb{C}$-algebra, since $\mathrm{char}(\mathbb{C}) = 0$.

It turns out that the converse to Maschke's theorem also holds. This follows from elementary properties of the augmentation ideal.

Theorem 6.2 (*Converse of Maschke's Theorem*)

If KG is a semisimple K-algebra, then $\mathrm{char}(K) \nmid |G|$.

This result can be proved using the Artin-Wedderburn Theorem and elementary properties of augmentation ideal through the following exercices.

Exercise 6.3 (*The augmentation ideal*)

The map $\varepsilon : KG \longrightarrow K, \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \lambda_g$ is an algebra homomorphism, called **augmentation homomorphism (or map)**. Its kernel $\ker(\varepsilon) =: I(KG)$ is an ideal, called the **augmentation ideal** of KG. Prove that:

(a) $I(KG) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\} = \mathrm{ann}_{KG}(K)$ and $I(KG) \supseteq J(KG)$;

(b) $KG/I(KG) \cong K$ as K-algebras;

(c) $I(KG)$ is a free K-vector space of dimension $|G|$-1 with K-basis $\{g - 1 \mid g \in G\backslash\{1\}\}$.

Exercise 6.4 (***Proof of the Converse of Maschke's Theorem.***)

Assume K is a field of positive characteristic p with $p \mid |G|$. Set $T := \langle \sum_{g \in G} g \rangle_K$.

(a) Prove that we have a series of KG-submodules given by $KG^\circ \supsetneq I(KG) \supseteq T \supsetneq 0$.

(b) Deduce that KG° has at least two composition factors isomorphic to the trivial module K.

(c) Deduce that KG is not a semisimple K-algebra.

Corollary 6.5

If $\mathrm{char}(K) \nmid |G|$, then $|G| = \sum_{S \in \mathrm{Irr}(KG)} \dim_K(S)^2$.

Proof: Since $\mathrm{char}(K) \nmid |G|$, the group algebra KG is semisimple by Maschke's Theorem. Thus

$$\sum_{S \in \mathrm{Irr}(KG)} \dim_K(S)^2 = \dim_K(KG) = |G| .$$

■

7 Clifford Theory

We now turn to *Clifford's theorem*, which we present in a weak and a strong form. Broadly speaking, Clifford theory is a collection of results about induction and restriction of simple modules from/to normal subgroups.

Theorem 7.1 (***Clifford's Theorem, weak form***)

If $U \trianglelefteq G$ is a normal subgroup and S is a simple KG-module, then $S{\downarrow}_U^G$ is semisimple.

Theorem 7.2 (***Clifford's Theorem, strong form***)

Let $U \trianglelefteq G$ be a normal subgroup and let S be a simple KG-module. Then we may write

$$S{\downarrow}_U^G = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$$

where $r \in \mathbb{Z}_{>0}$ and $S_1, \ldots, S_r$ are pairwise non-isomorphic simple KU-modules, occurring with multiplicities $a_1, \ldots, a_r$ respectively. Moreover, the following statements hold:

(i) the group G permutes the homogeneous components of $S{\downarrow}_U^G$ transitively;

(ii) $a_1 = a_2 = \cdots = a_r$ and $\dim_K(S_1) = \cdots = \dim_K(S_r)$; and

(iii) $S \cong (S_1^{a_1}){\uparrow}_{H_1}^G$ as KG-modules, where $H_1 = \mathrm{Stab}_G(S_1^{a_1})$.

One application of Clifford's theory is for example the following Corollary:

Corollary 7.3

Assume K is a field of arbitrary characteristic. (Still splitting for G.) If p is a prime number and G is a p-group, then every simple KG-module has the form $X{\uparrow}_H^G$, where X is a 1-dimensional KH-module for some subgroup $H \leqslant G$.

Remark 7.4

This result is extremely useful, for example, to construct the complex character table of a p-group. Indeed, it says that we need look no further than induced linear characters. In general, a KG-module of the form $N{\uparrow}_H^G$ for some subgroup $H \leqslant G$ and some 1-dimensional KH-module is called **monomial**. A group all of whose simple $\mathbb{C}G$-modules are monomial is called an M-**group**. (By the above p-groups are M-groups.)

After simple and semisimple modules, the goal of this chapter is to understand indecomposable modules in general. Apart for exceptions, the group algebra is of *wild representation type*, which, roughly speaking, means that it is not possible to classify the indecomposable modules over such algebras. However, representation theorists have developed tools which enable us to organise indecomposable modules in packages parametrised by parameters that are useful enough to understand essential properties of these modules. In this respect, we will generalise the idea of a projective module by defining what is called **relative projectivity**. This will lead us to introduce the concepts of **vertices** and **sources** of indecomposable modules, which are two typical examples of parameters bringing us useful information about indecomposable modules in general.

8 Existence and Uniqueness of Direct Sum Decompositions

First, we take a look at the concept of decomposability over general rings.

Definition 8.1 (*indecomposable module*)

An R-module M is called **decomposable** if M possesses two non-zero proper submodules M_1, M_2 such that $M = M_1 \oplus M_2$. An R-module M is called **indecomposable** if it is non-zero and not decomposable.

First, we want to be able to decompose R-modules into direct sums of indecomposable submodules. The Krull-Schmidt Theorem then provide us with certain uniqueness properties of such decompositions.

Proposition 8.2

Let M be an R-module. If M satisfies either A.C.C. or D.C.C., then M admits a decomposition into a direct sum of finitely many indecomposable R-submodules.

Exercise 8.3

Prove Proposition 8.2.

Theorem 8.4 (*Krull–Schmidt*)

Let M be an R-module which has a composition series. If

$$M = M_1 \oplus \cdots \oplus M_n = M_1' \oplus \cdots \oplus M_{n'}' \qquad (n, n' \in \mathbb{Z}_{>0})$$

are two decomposition of M into direct sums of finitely many indecomposable R-submodules, then $n = n'$, and there exists a permutation $\pi \in \mathfrak{S}_n$ such that $M_i \cong M'_{\pi(i)}$ for each $1 \leqslant i \leqslant n$.

Thus the number n is uniquely determined by the module M, and the submodules $M_1, \ldots, M_n$ are unique, up to isomorphism and ordering. They are sometimes called the **components** of M.

9 Indecomposability Criteria

The proof of the Krull-Schmidt theorem relies on the following general indecomposability criterion.

Proposition 9.1 (*Indecomposability criterion*)

Let M be an R-module which has a composition series. Then:

$$M \text{ is indecomposable } \iff \mathrm{End}_R(M) \text{ is a local ring.}$$

For modules over the group algebra, we have the following important indecomposability criterion due to J. A. Green. The proof is rather involved.

Theorem 9.2 (*Green's indecomposability criterion, 1959*)

Assume that K is an algebraically closed field of characteristic $p > 0$. Let $H \leqslant G$ be a subnormal subgroup of G of index a power of p and let M be an indecomposable KH-module. Then $M\!\uparrow_H^G$ is an indecomposable KG-module.

Remark 9.3

Green's indecomposability criterion remains true over an arbitrary field of characteristic p, provided we replace *indecomposability* with *absolute indecomposability*. (A KG-module M is called absolutely indecomposable iff its endomorphism algebra $\mathrm{End}_{KG}(M)$ is a *split local algebra*, that is, if $\mathrm{End}_{KG}(M)/J(\mathrm{End}_{KG}(M)) \cong K$.)

Example 5

Assume that K is an algebraically closed field of characteristic $p > 0$. If P is a p-group, $Q \leqslant P$ and M is an indecomposable KQ-module, then $M\!\uparrow_Q^P$ is an indecomposable KP-module. In particular, the permutation module $K[P/Q] \cong K\!\uparrow_Q^P$ is indecomposable.

Indeed, since P is a p-group, by the Sylow theory any subgroup $Q \leqslant P$ can be plugged in a subnormal series where each quotient is cyclic of order p, hence is a subnormal subgroup of P. The claim follows immediately from Green's indecomposability criterion.

10 Projective Modules for the Group Algebra

We have seen that over a semisimple ring, all simple modules appear as direct summands of the regular module with multiplicity equal to their dimension. For non-semisimple rings this is not true any more, but replacing simple modules by the *projective* modules, we will obtain a similar characterisation.

To begin with we review a series of properties of projective KG-modules with respect to the operations on groups and modules we have introduced in Chapter 1, i.e. induction/restriction, tensor products, ...

Proposition 10.1

Assume K is an arbitrary commutative ring. Then the following assertions hold.

(a) If P is a projective KG-module and M is an arbitrary KG-module, then $P \otimes_K M$ is projective.

(b) If P is a projective KG-module and $H \leqslant G$, then $P{\downarrow}_H^G$ is a projective KH-module.

(c) If $H \leqslant G$, then $KH{\uparrow}_H^G \cong KG$ and P is a projective KH-module, then $P{\uparrow}_H^G$ is a projective KG-module. [Hint: Prove that $KH{\uparrow}_H^G \cong KG$.]

Exercise 10.2

Prove Proposition 10.1.

We now want to prove that the PIMs of KG can be labelled by the simple KG-modules, and hence that there are a finite number of them, up to isomorphism. We will then be able to deduce from this bijection that each of them occurs in the decomposition of the regular module with multiplicity equal to the dimension of the corresponding simple module.

Theorem 10.3

(a) If P is a projective indecomposable KG-module, then $P/\operatorname{rad}(P)$ is a simple KG-module.

(b) If M is a KG-module and $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ for a projective indecomposable KG-module P, then there exists a surjective KG-homomorphism $\varphi : P \longrightarrow M$. In particular, if M is also projective indecomposable, then $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ if and only if $M \cong P$.

(c) There is a bijection

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{projective indecomposable}\\ KG\text{-modules}\end{array}\right\}/\cong & \stackrel{\sim}{\longleftrightarrow} & \left\{\begin{array}{c}\text{simple}\\ KG\text{-modules}\end{array}\right\}/\cong \\ P & \mapsto & P/\operatorname{rad}(P) \end{array}$$

and hence the number of pairwise non-isomorphic PIMs of KG is finite.

Definition 10.4 (*Projective cover of a simple module*)

If S is a simple KG-module, then we denote by P_S the projective indecomposable KG-module corresponding to S through the bijection of Theorem 10.3(c) and call this module the **projective cover** of S.

Corollary 10.5

In the decomposition of the regular module KG into a direct sum of indecomposable KG-submodules, each isomorphism type of projective indecomposable KG-module occurs with multiplicity $\dim_K(P/\operatorname{rad}(P))$. In other words,

$$KG \cong \bigoplus_{S\in\operatorname{Irr}(KG)} (P_S)^{n_s}$$

(where $n_S = \dim_K S$).

Proof: Let $KG = P_1 \oplus \cdots \oplus P_r$ ($r \in \mathbb{Z}_{>0}$) be such a decomposition. In particular, $P_1, \ldots P_r$ are PIMs. Then

$$J(KG) = J(KG)KG = J(KG)P_1 \oplus \cdots \oplus J(KG)P_r = \operatorname{rad}(P_1) \oplus \cdots \oplus \operatorname{rad}(P_r)\,.$$

Therefore,

$$KG/J(KG) \cong P_1/\operatorname{rad}(P_1) \oplus \cdots \oplus P_r/\operatorname{rad}(P_r)$$

where each summand is simple by Theorem 10.3(a). Now as $KG/J(KG)$ is semisimple, by Theorem 5.5, any simple $KG/J(KG)$-module occurs in this decomposition with multiplicity equal to its K-dimension. Thus the claim follows from the bijection of Theorem 10.3(c). ■

The Theorem also leads us to the following important dimensional restriction on projective modules, which we will see again later.

Exercise 10.6

Assume K is a splitting field for G of characteristic $p > 0$.

(a) Prove that if G is a p-group, then the projective cover of the trivial module is the regular module.

(b) Use (a) and restriction to a Sylow p-subgroup to prove that if P is a projective KG-module, then

$$|G|_p \mid \dim_K(P)\,.$$

(Here $|G|_p$ is the p-part of $|G|$, i.e. the exact power of p that divides the order of G.)

11 Relative Projectivity

Definition 11.1

Let $H \leqslant G$. A KG-module M is called **relatively H-projective**, or simply **H-projective**, if it is isomorphic to a direct summand of a KG-module induced from H, i.e. if $M \mid V\uparrow_H^G$ for some KH-module V.

Example 6

Clearly, H-projectivity is a generalisation of projectivity. Indeed, if $M \in \operatorname{mod}(KG)$, then:

$$\begin{aligned} M \text{ is projective} &\iff \exists n \in \mathbb{Z}_{>0} \text{ such that } M \mid (KG)^n \cong (K\uparrow_{\{1\}}^G)^n \cong (K^n)\uparrow_{\{1\}}^G \\ &\iff M \text{ is } \{1\}\text{-projective} \end{aligned}$$

We can actually characterise relative projectivity in a similar way as we characterised projectivity.

Proposition 11.2 (*Characterisation of relative projectivity*)

Let $H \leqslant G$ and let M be a KG-module. TFAE:

(a) M is relatively H-projective;

(b) $M \mid M{\downarrow}_H^G{\uparrow}_H^G$;

(c) $\exists$ a KG-module N such that $M \mid K{\uparrow}_H^G \otimes_K N$;

(d) if $\psi \in \mathrm{Hom}_{KG}(M, W)$, $\varphi \in \mathrm{Hom}_{KG}(V, W)$ is surjective and $\exists\ \alpha_H \in \mathrm{Hom}_{KH}(M{\downarrow}_H^G, V{\downarrow}_H^G)$ such that $\varphi \circ \alpha_H = \psi$, then $\exists\ \alpha_G \in \mathrm{Hom}_{KG}(M, V)$ such that $\varphi \circ \alpha_G = \psi$;

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists\,\alpha_G"'] \arrow[d, "\psi"] \\ V \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft"] & W \end{tikzcd}$$

(e) A surjective KG-homomorphism $\varphi : V \twoheadrightarrow M$ is KG-splits provided it is KH-split.

Projectivity relative to a subgroup can be generalised as follows to projectivity relative to a KG-module:

Remark 11.3 (*Projectivity relative to KG-modules*)

(a) Let V be a KG-module. A KG-module M is termed *projective relative to the module V* or *relatively V-projective*, or simply *V-projective* if there exists a KG-module N such that M is isomorphic to a direct summand of $V \otimes_K N$, i.e. $M \mid V \otimes_K N$.

(b) Proposition 11.2(c) shows that projectivity relative to a subgroup $H \leqslant G$ is in fact projectivity relative to the KG-module $V := K{\uparrow}_H^G$.

The concept of projectivity relative to a subgroup is proper to the group algebra, but the concept of projectivity relative to a module is not and makes sense in general over algebras/rings.

Next we see that any indecomposable KG-module can be seen as a relatively projective module with respect to some subgroup of G.

Theorem 11.4

Let $H \leqslant G$.

(a) If $|G : H|$ is invertible in K, then every KG-module is H-projective.

(b) In particular, if K is a field of characteristic $p > 0$ and H contains a Sylow p-subgroup of G, then every KG-module is H-projective.

Part (b) follows immediately from (a). Indeed, if $P \in \mathrm{Syl}_p(G)$ and $H \supseteq P$, then $p \nmid |G : H|$, so $|G : H| \in K^\times$. Moreover, considering the case $H = \{1\}$ shows that Theorem 11.4 is a generalisation of Maschke's Theorem.

Example 7

Assume that $\mathrm{char}(K) =: p > 0$ and $H = \{1\}$. If H contains a Sylow p-subgroup of G then the Sylow p-subgroups of G are trivial, so $p \nmid |G|$. The theorem then says that all KG-modules are $\{1\}$-projective, that is, projective.

We know this already, however! If $p \nmid |G|$ then KG is semisimple by Maschke's Theorem, and so all KG-modules are projective.

Corollary 11.5

Let $H \leqslant G$ and suppose that $|G : H|$ is invertible in K. Then a KG-module M is projective if and only if $M\downarrow_H^G$ is projective.

Proof: The necessary condition is given by Proposition 10.1(b). To prove the sufficient condition, suppose that $M\downarrow_H^G$ is projective. Then, on the one hand,

$$M\downarrow_H^G \mid (KH)^n \qquad \text{for some } n \in \mathbb{Z}_{>0}\,.$$

On the other hand, M is H-projective by Theorem 11.4, and it follows from Proposition 11.2(e) that

$$M \mid M\downarrow_H^G\uparrow_H^G\ .$$

Hence

$$M \mid M\downarrow_H^G\uparrow_H^G \mid (KH)^n\uparrow_H^G \cong (KG)^n\,,$$

so M is projective. ■

12 Vertices and Sources

As in the case in which KG is semisimple, relative projectivity is just projectivity, we now focus on the non-semiminple case.

For the remainder of this chapter, we assume that $\mathrm{char}(K) =: p > 0$ and $p \mid |G|$.

As said before, we now want to explain some techniques that are available to understand indecomposable modules better. Vertices and sources are two parameters making this possible.

Theorem 12.1

Let M be an indecomposable KG-module.

(a) There is a unique conjugacy class of subgroups Q of G which are minimal subject to the property that M is Q-projective.

(b) Let Q be a minimal subgroup of G such that M is Q-projective. Then, there exists an indecomposable KQ-module T which is unique, up to conjugacy by elements of $N_G(Q)$, such that M is a direct summand of $T\uparrow_Q^G$. Such a KQ-module T is necessarily a direct summand of $M\downarrow_Q^G$.

This characterisation leads us to the following definition:

Definition 12.2

Let M be an indecomposable KG-module.

(a) A **vertex** of M is a minimal subgroup Q of G such that M is relatively Q-projective. The set of all vertices of M is denoted by $\mathrm{vtx}(M)$.

(b) Given a vertex Q of M, a **KQ-source**, or simply a **source** of M is a KQ-module T such that $M \mid T\uparrow_Q^G$.

Remark 12.3

(a) A vertex Q of an indecomposable KG-module M is not uniquely defined, in general. However, the vertices of M are unique **up to G-conjugacy**, so in particular are all isomorphic. For this reason, in general, one (i.e. you!) should **never** talk about *the* vertex of a module (of course, unless a vertex has been fixed). We either say that Q *is a vertex of* M, or talk about *the vertices of* M. (Unfortunately many textbooks/articles are very sloppy with this terminology, inducing errors.)

(b) For a fixed vertex Q of M, a source of M is defined up to conjugacy by elements of $N_G(Q)$.

Warning! Vertices and sources are very useful theoretical tools in general, but extremely difficult to compute concretely. However, the following properties are useful.

To begin with, by Theorem 11.4, we know that every KG-module is projective relative to a Sylow p-subgroup of G. Therefore, by minimality, vertices are contained in Sylow p-subgroups. Hence:

Proposition 12.4

The vertices of an indecomposable KG-module are p-subgroups of G.

Proposition 12.5

Let U be an indecomposable KG-module and let $Q \in \mathrm{vtx}(U)$. If $P \in \mathrm{Syl}_p(G)$ is such that $Q \subseteq P$, then

$$|P : Q| \mid \dim_K(U)\,.$$

In particular if U is a PIM of KG, then $|P| = |G|_p \mid \dim_K(U)$.

Example 8

(a) The trivial subgroup $\{1\}$ is a vertex of an indecomposable KG-module $U \iff U$ is a PIM of KG.

(b) The vertices of the trivial KG-module are the Sylow p-subgroups of G, i.e. $\mathrm{vtx}(K) = \mathrm{Syl}_p(G)$, and all sources are trivial.

Exercise 12.6

Prove that the vertices of any KG-module with K-dimension coprime to p are the Sylow p-subgroups of G.

Conceptually, the closer the vertices of a module are to the trivial subgroup, the closer this module is to being projective.

Finally, we give a name to the modules which have a trivial source. We will see in Lecture 4 that these module play a particularly important role in block theory.

Definition 12.7 (*trivial source module*)

A KG-module is called a **trivial source** KG-module if it is a finite direct sum of KG-modules with a trivial source K.

Warning! Some texts (books/articles/...) require that a trivial source module is indecomposable, others do not.

13 The Green Correspondence

The Green correspondence is a correspondence which relates the indecomposable KG-modules with a fixed vertex with the indecomposable KL-modules with the same vertex for well-chosen subgroups $L \leqslant G$. It is used to reduce questions about indecomposable modules to a situation where a vertex of the given indecomposable module is a normal subgroup.

Theorem 13.1 (*Green Correspondence*)

Let Q be a p-subgroup of G and let L be a subgroup of G containing $N_G(Q)$.

(a) If U is an indecomposable KG-module with vertex Q, then

$$U\downarrow_L^G = f(U) \oplus X$$

where $f(U)$ is the unique indecomposable direct summand of $U\downarrow_L^G$ with vertex Q and every direct summand of X is $L \cap {}^xQ$-projective for some $x \in G \backslash L$.

(b) If V is an indecomposable KL-module with vertex Q, then

$$V\uparrow_L^G = g(V) \oplus Y$$

where $g(V)$ is unique indecomposable direct summand of $V\uparrow_L^G$ with vertex Q and every direct summand of Y is $Q \cap {}^xQ$-projective for some $x \in G \backslash L$.

(c) With the notation of (a) and (b), we then have $g(f(U)) \cong U$ and $f(g(V)) \cong V$. In other words, f and g define a bijection

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{isomorphism classes of indecomposable}\\ KG\text{-modules with vertex } Q\end{array}\right\} & \xleftrightarrow{\sim} & \left\{\begin{array}{c}\text{isomorphism classes of indecomposable}\\ KL\text{-modules with vertex } Q\end{array}\right\} \\ U & \mapsto & f(U) \\ g(V) & \leftarrow\!\shortmid & V\ . \end{array}$$

Moreover, corresponding modules have a source in common.

Terminology: $f(U)$ is called the KL-**Green correspondent** of U (or simply the **Green correspondent**) and $g(V)$ is called the KG-**Green correspondent of** V (or simply the **Green correspondent** of V).

Warning! When working with the Green correspondence it is essential that a vertex Q is fixed and not considered up to conjugation, because the G-conjugacy class of Q and the L-conjugacy class of Q do not coincide in general.

Example 9

The Green correspondent of the trivial module is the trivial module, for $K\downarrow_L^G = K$.

14 p-Permutation Modules

Definition 14.1 (*Permutation module*)

A KG-module is called a **permutation** KG-module if it admits a K-basis X which is invariant under the action of the group G. We denote this module by KX.

Permutation KG-modules and, in particular, their indecomposable direct summands have remarkable properties, which we investigate in this section.

Remark 14.2

If KX is a permutation KG-module on X, then a decomposition of the basis X as a disjoint union of G-orbits, say $X = \bigsqcup_{i=1}^n X_i$, yields a direct sum decomposition of KX as a KG-module as

$$KX = \bigoplus_{i=1}^{n} KX_i\,.$$

Thus, we can assume that X is a transitive G-set, in which case we have a direct sum decomposition as a K-vector space

$$KX = \bigoplus_{g\in[G/H]} Kgx$$

where $H := \mathrm{Stab}_G(x)$, the stabiliser in G of some $x \in X$, and G acts transitively on the summands. Hence,

$$KX \cong K\uparrow_H^G\,.$$

It follows that an arbitrary permutation KG-module is isomorphic to a direct sum of KG-modules of the form $K\uparrow_H^G$ for various $H \leqslant G$.

Conversely, an induced module of the form $K\uparrow_H^G$ ($H \leqslant G$) is always a permutation KG-module. Indeed, as $K\uparrow_H^G = KG \otimes_{KH} K = \bigoplus_{g\in[G/H]} g \otimes K$ as K-vector space, it has on obvious G-invariant K-basis given by the set

$$\{g \otimes 1_K \mid g \in [G/H]\}\,.$$

In fact, more generally if $H \leqslant G$ and KX is a permutation KH-module on X, then $KX\uparrow_H^G$ is a permutation KG-module with G-invariant K-basis $\{g \otimes x \mid g \in [G/H], x \in X\}$. In other words, induction preserves permutation modules.

Exercise 14.3

Prove that direct sums, restriction, inflation and conjugation also preserve permutation modules.

In order to understand the indecomposable direct summands of the permutation KG-modules, we observe that they all have a trivial source and we will apply the Green correspondence to see that, up to isomorphism, there are only a finite number of them.

Proposition-Definition 14.4 (*p-permutation module*)

Let M be a KG-module and let $P \in \mathrm{Syl}_p(G)$. Then, the following conditions are equivalent:

(a) $M\downarrow_Q^G$ is a permutation KQ-module for each p-subgroup $Q \leqslant G$;

(b) $M\downarrow_P^G$ is a permutation KP-module;

(c) M has a K-basis which is invariant under the action of P;

(d) M is isomorphic to a direct summand of a permutation KG-module;

(e) M is a trivial source KG-module.

If M fulfils one of these equivalent conditions, then it is called a ***p*-permutation** KG-module.

Note. In fact *p-permutation* KG-modules and *trivial source* KG-modules are two different pieces of terminology for the same concept. French/German speaking authors tend to favour the terminology *p-permutation module* (and reserve the terminology *trivial source module* for an indecomposable module with a trivial source), whereas English speaking authors tend to favour the terminology *trivial source module*.

Exercise 14.5

Prove that p-permutation modules are preserved by the following operations: direct sums, tensor products, restriction, inflation, conjugation, induction.

Example 10

(a) If G is a p-group, then any p-permutation module is a permutation module.

(b) The PIMs of KG are precisely the KG-modules with vertex $\{1\}$ and trivial source, so any projective KG-module is a p-permutation KG-module.

Example 11

Any KG-modules Y of K-dimension 1 is a p-permutation module.
Proof. Let Q be a vertex of Y and let $f(Y)$ be the $kN_G(Q)$-Green correspondent of Y. Then clearly $\dim_K f(Y) = 1$ as well. Thus $f(Y)$ is a simple and therefore has a trivial KQ-source. Indeed, $f(Y)\downarrow_Q^{N_G(Q)}$ is semisimple by the weak version of Clifford's Theorem, and so must be a direct sum of copies of the trivial kQ-module because Q is a p-group and therefore has only one simple module, up to isomorphism, namely the trivial module.

Generalising these examples, we can characterise the indecomposable p-permutation KG-modules with a given vertex $Q \leqslant G$ as described below.

Example 12 (*Green Correspondence applied to indecomposables with a trivial source*)

(1) If M is an indecomposable p-permutation KG-module with vertex $Q \leqslant G$, then Q acts trivially on the $KN_G(Q)$-Green correspondent $f(M)$ of M. Thus $f(M)$ can be viewed as a $K[N_G(Q)/Q]$-module. As such, $f(M)$ is indecomposable and projective.

(2) Conversely, if N is a projective indecomposable $K[N_G(Q)/Q]$-module, then $\mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(N)$ is an indecomposable $KN_G(Q)$-module with vertex Q and trivial source. Its KG-Green correspondent is then also an indecomposable KG-module with vertex Q and trivial source, hence is an indecomposable p-permutation KG-module

(3) In this way we obtain a bijection

$$\left\{\begin{matrix}\text{isomorphism classes of indecomposable}\\ p\text{-permutation } KG\text{-modules with vertex } Q\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{isomorphism classes of projective}\\ \text{indecomposable } K[N_G(Q)/Q]\text{-modules}\end{matrix}\right\}.$$

Wednesday. Chapter 4. p-Modular Systems and Brauer Characters

R. Brauer started in the late 1920's a systematic investigation of group representations over fields of positive characteristic. In order to relate group representations over fields of positive characteristic to character theory in characteristic zero, Brauer worked with a triple of rings $(F, \mathcal{O}, k)$, called a *p-modular system*, and consisting of a complete discrete valuation ring $\mathcal{O}$ with a residue field $k := \mathcal{O}/J(\mathcal{O})$ of prime characteristic p and fraction field $F := \mathrm{Frac}(\mathcal{O})$ of characteristic zero. These are used to gain information about kG and its modules (which is/are extremely complicated) from the group algebra FG, which is semisimple and therefore much better understood, via the group algebra $\mathcal{O}G$. This explains why we considered arbitrary associative rings (resp. algebras / fields) in the previous chapters rather than immediately focusing on fields of positive characteristic.

Notation. Throughout this chapter, unless otherwise specified, we let p be a prime number and let $\Lambda \in \{F, \mathcal{O}, k\}$.

15 p-Modular Systems

Recall that a commutative ring $\mathcal{O}$ is local iff $\mathcal{O} \setminus \mathcal{O}^\times = J(\mathcal{O})$, i.e. $J(\mathcal{O})$ is the unique maximal ideal of $\mathcal{O}$. Moreover, by the commutativity assumption this happens if and only if $\mathcal{O}/J(\mathcal{O})$ is a field. In such a situation, we write $k := \mathcal{O}/J(\mathcal{O})$ and call this field **the residue field of the local ring** $\mathcal{O}$. To ease up notation, we will often write $\mathfrak{p} := J(\mathcal{O})$. This is because our aim is a situation in which the residue field is a field of positive characteristic p. Moreover, a commutative ring $\mathcal{O}$ is called a **discrete valuation ring** if $\mathcal{O}$ is a local principal ideal domain such that $J(\mathcal{O}) \neq 0$. Such a discrete valuation ring is called **complete** if it is complete in the $J(\mathcal{O})$-adic topology.

Definition 15.1 (*p-modular systems*)

Let p be a prime number.

(a) A triple of rings $(F, \mathcal{O}, k)$ is called a **p-modular system** if:

  (1) $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero,

  (2) $F = \mathrm{Frac}(\mathcal{O})$ is the field of fractions of $\mathcal{O}$ (also of characteristic zero), and

  (3) $k = \mathcal{O}/J(\mathcal{O})$ is the residue field of $\mathcal{O}$ and has characteristic p.

(b) If G is a finite group, then a p-modular system $(F, \mathcal{O}, k)$ is called a **splitting p-modular system for** G, if both F and k are splitting fields for G.

It is often helpful to visualise p-modular systems and the condition on the characteristic of the rings involved through the following commutative diagram of rings and ring homomorphisms:

$$\begin{tikzcd} \mathbb{Q} \arrow[d, hook] & \mathbb{Z} \arrow[l, hook'] \arrow[r, two heads] \arrow[d, hook] & \mathbb{F}_p \arrow[d, hook] \\ F & \mathcal{O} \arrow[l, hook'] \arrow[r, two heads] & k \end{tikzcd}$$

where the hook arrows are the canonical inclusions and the two-head arrows the quotient morphisms. Clearly, these morphisms also extend naturally to ring homomorphisms

$$FG \hookleftarrow \mathcal{O}G \twoheadrightarrow kG$$

between the corresponding group algebras (each mapping an element $g \in G$ to itself).

Example 13

One usually works with a splitting p-modular system for all subgroups of G, because it allows us avoid problems with field extensions. By a theorem of Brauer on splitting fields such a p-modular system can always be obtained by adjoining a primitive m-th root of unity to $\mathbb{Q}_p$, where m is the exponent of G. (Notice that this extension is unique.) So we may as well assume that our situation is as given in the following commutative diagram:

$$\begin{tikzcd} \mathbb{Q}_p \arrow[d, hook] & \mathbb{Z}_p \arrow[l, hook'] \arrow[r, two heads] \arrow[d, hook] & \mathbb{F}_p \arrow[d, hook] \\ F & \mathcal{O} \arrow[l, hook'] \arrow[r, two heads] & k \end{tikzcd}$$

More generally, we have the following result, which we mention without proof. The proof can be found in §17A of Volume 1 of Curtis and Reiner's book.

Theorem 15.2

Let $(F, \mathcal{O}, k)$ be a p-modular system. Let G be a finite group of exponent $m := exp(G)$. Then the following assertions hold.

(a) The field F contains all m-th roots of unity if and only if F contains the cyclotomic field of m-th roots of unity;

(b) If F contains all m-th roots of unity, then so does k and F and k are splitting fields for G and all its subgroups.

Remark 15.3

If $(F, \mathcal{O}, k)$ is a p-modular system, then it is not possible to have F and k algebraically closed, while assuming $\mathcal{O}$ is complete. (Depending on your knowledge on discrete valuation rings, you can try to prove this as an exercise!)

Let us now investigate changes of the coefficients given in the setting of a p-modular system for group algebras involved.

Definition 15.4

Let $\mathcal{O}$ be a commutative local ring. A finitely generated $\mathcal{O}G$-module L is called an **$\mathcal{O}G$-lattice** if it is free (= projective) as an $\mathcal{O}$-module.

Remark 15.5 (*Changes of the coefficients*)

Let $(F, \mathcal{O}, k)$ be a p-modular system and write $\mathfrak{p} := J(\mathcal{O})$. If L is an $\mathcal{O}G$-module, then:

- setting $L^F := F \otimes_{\mathcal{O}} L$ defines an FG-module, and
- reduction modulo $\mathfrak{p}$ of L, that is $\overline{L} := L/\mathfrak{p}L \cong k \otimes_{\mathcal{O}} L$ defines a kG-module.

We note that, when seen as an $\mathcal{O}$-module, an $\mathcal{O}G$-module L may have torsion, which is lost on passage to F. In order to avoid this issue, we usually only work with $\mathcal{O}G$-lattices. In this way, we obtain functors

$$FG\text{-}\mathbf{mod} \longleftarrow \mathcal{O}G\text{-}\mathbf{lat} \longrightarrow\!\!\!\!\rightarrow kG\text{-}\mathbf{mod}$$

between the corresponding categories of finitely generated $\mathcal{O}G$-lattices and finitely generated FG-, kG-modules.

A natural question to ask is: which FG-modules, respectively kG-modules, come from $\mathcal{O}G$-lattices? In the case of FG-modules we have the following answer.

Proposition-Definition 15.6

Let $\mathcal{O}$ be a complete discrete valuation ring and let $F := \mathrm{Frac}(\mathcal{O})$ be the fraction field of $\mathcal{O}$. Then, for any finitely generated FG-module V there exists an $\mathcal{O}G$-lattice L which has an $\mathcal{O}$-basis which is also an F-basis. In this situation $V \cong L^F$ and we call L an **$\mathcal{O}$-form** of V.

Proof: Choose an F-basis $\{v_1, \ldots, v_n\}$ of V and set $L := \mathcal{O}Gv_1 + \cdots + \mathcal{O}Gv_n \subseteq V$. ■

On the other hand, the question has a negative answer for kG-modules.

Definition 15.7 (*liftable kG-module*)

Let $\mathcal{O}$ be a commutative local ring with unique maximal ideal $\mathfrak{p} := J(\mathcal{O})$ and residue field $k := \mathcal{O}/\mathfrak{p}$. A kG-module M is called **liftable** if there exists an $\mathcal{O}G$-lattice $\widehat{M}$ whose reduction modulo $\mathfrak{p}$ of M is isomorphic to M, that is

$$\widehat{M}/\mathfrak{p}\widehat{M} \cong M\,.$$

(Alternatively, it is also said that M is **liftable to an $\mathcal{O}G$-lattice**, or **liftable to $\mathcal{O}$**, or **liftable to characteristic zero**.)

Even though every $\mathcal{O}G$-lattice can be reduced modulo $\mathfrak{p}$ to produce a kG-module, not every kG-module is liftable to an $\mathcal{O}G$-lattice.

Being liftable for a kG-module is a rather rare property. However, some classes of kG-modules do lift.

Example 14

It follows from the *lifting of idempotents* theorem that projective indecomposable kG-modules are liftable to projective indecomposable $\mathcal{O}G$-lattices:

Any (projective) indecomposable kG-module is liftable to a (projective) indecomposable $\mathcal{O}G$-lattice. More generally, any trivial source kG-module M is liftable to an $\mathcal{O}G$-lattice. More precisely, among all lifts of M a unique one is again trivial source and we denote it by $\widetilde{M}$.
The F-character of $F \otimes_{\mathcal{O}} \widetilde{M}$ is called **the ordinary character of** M.

16 Brauer Characcters

Recall that we have fixed a splitting p-modular system $(F, \mathcal{O}, k)$ such that F contains an $\exp(G)$-th root of unity. Since F is a field of characteristic zero, FG-modules are isomorphic if and only if their characters are equal. Also, the character of an FG-module provides complete information about its composition factors, including multiplicities, provided that the irreducible characters of G are known. All this does not hold for fields k of characteristic $p > 0$. For instance, if W is a k-vector space on which G acts trivially and $\dim_k(W) = ap + 1$ for some non-negative integer a, then the k-character of W is the trivial character. This implies that a k-character can only give information about multiplicities of composition factors modulo p. In view of these issues, the aim of this chapter is to define a slightly different kind of *character theory* for modular representations of finite groups and to establish links with ordinary character theory.

Recall that an element $g \in G$ is called a p**-regular element** (or a p'**-element**) if $p \nmid o(g)$. We write

$$G_{p'} := \{g \in G \mid p \nmid o(g)\}$$

for the set of all p-regular elements of G.

Since F contains all $\exp(G)$-th roots of unity, both F and k contain a primitive a-th root of unity, where a is the l.c.m. of the orders of the p-regular elements. Set

$$\mu_F := \{a\text{-th roots of 1 in } F\} \text{ and } \mu_k := \{a\text{-th roots of 1 in } k\}.$$

Then $\mu_F \subseteq \mathcal{O}$ and, as both μ_F and μ_k are finite groups, it follows that the quotient morphism $\mathcal{O} \twoheadrightarrow \mathcal{O}/\mathfrak{p}$ restricted to μ_F induces a group isomorphism

$$\mu_F \xrightarrow{\cong} \mu_k .$$

We write the underlying bijection as $\widehat{\xi} \mapsto \xi$, so that if ξ is an a-th root of unity in k then $\widehat{\xi}$ is the unique a-th root of unity in $\mathcal{O}$ which maps onto it.

Lemma 16.1 (*Diagonalisation lemma*)

Let $\rho : G \longrightarrow \mathrm{GL}(U)$ be a k-representation of G. Then, for every p-regular element $g \in G_{p'}$, the k-linear map $\rho(g)$ is diagonalisable and the eigenvalues of $\rho(g)$ are $o(g)$-th roots of unity and lie in μ_k. In other words, there exists an ordered k-basis B of U with respect to which

$$\big(\rho(g)\big)_B = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi_n \end{bmatrix},$$

where $n := \dim_k(U)$ and each ξ_i ($1 \leqslant i \leqslant n$) is an $o(g)$-th root of unity in k.

Proof: Let $g \in G_{p'}$. It is enough to consider the restriction of ρ to the cyclic subgroup $\langle g \rangle$. Since $p \nmid |\langle g \rangle|$, $k\langle g \rangle$ is semisimple by Maschke's Theorem. Moreover, as k is a splitting field for $\langle g \rangle$, it follows from Corollary 5.10 that all irreducible k-representations of $\langle g \rangle$ have degree 1. Hence $\rho|_{\langle g \rangle}$ can be decomposed as the direct sum of degree 1 subrepresentations. As a consequence $\rho(g) = \rho|_{\langle g \rangle}(g)$ is diagonalisable and there exists a k-basis B of U satisfying the statement of the lemma. It follows immediately that the eigenvalues are $o(g)$-th roots of unity because $\rho_U(g^{o(g)}) = \rho_U(1_G) = \mathrm{Id}_U$. They all lie in μ_k, being $o(g)$-th roots of unity, hence a-th roots of unity. ■

This leads to the following definition.

Definition 16.2 (*Brauer characters*)

Let U be a kG-module of dimension $n \in \mathbb{Z}_{\geqslant 0}$ and let $\rho_U : G \to \mathrm{GL}(U)$ be the associated k-representation. The p-**Brauer character** or simply the **Brauer character** of G afforded by U (resp. of ρ_U) is the F-valued function

$$\begin{aligned} \varphi_U : G_{p'} &\to \mathcal{O} \subseteq F \\ g &\mapsto \widehat{\xi}_1 + \cdots + \widehat{\xi}_n , \end{aligned}$$

where $\xi_1, \ldots, \xi_n \in \mu_k$ are the eigenvalues of $\rho_U(g)$. The integer n is also called the **degree** of φ_U. Moreover, φ_U is called **irreducible** if U is simple (resp. if ρ_U is irreducible), and it is called **linear** if $n = 1$. We denote by $\mathrm{IBr}_p(G)$ the set of all irreducible Brauer characters of G and we write $\mathbf{1}_{G_{p'}}$ for the Brauer character of the trivial kG-module.

In the sequel, we want to prove that Brauer characters of kG-modules have properties similar to $\mathbb{C}$-characters.

Remark 16.3

(a) **Warning**: $\varphi(g) \in \mathcal{O} \subseteq F$ even though $\rho_U(g)$ is defined over the field k of characteristic $p > 0$.

(b) Often the values of Brauer characters are considered as complex numbers, i.e. sums of complex roots of unity. Of course, in that case then $\varphi_U(g)$ depends on the choice of embedding of μ_F into $\mathbb{C}$. However, for a fixed embedding, $\varphi_U(g)$ is uniquely determined up to similarity of $\rho_U(g)$.

Immediate properties of Brauer characters are as follows.

Exercise 16.4

Let U, V, W be non-zero kG-modules. Prove the following assertions:

(a) $\varphi_U(1) = \dim_k(U)$.

(b) φ_U is a class function on $G_{p'}$.

(c) $\varphi_U(g^{-1}) = \varphi_{U^*}(g) \;\; \forall\, g \in G_{p'}$.

(d) If $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is a s.e.s. of kG-modules, then

$$\varphi_V = \varphi_U + \varphi_W\,.$$

(e) If the composition factors of U are $S_1, \ldots, S_m$ ($m \in \mathbb{Z}_{\geqslant 1}$) with multiplicities $n_1, \ldots, n_m$ respectively, then

$$\varphi_U = n_1\varphi_{S_1} + \ldots + n_m\varphi_{S_m}\,.$$

In particular, if two kG-modules have isomorphic composition factors, counting multiplicities, then they have the same Brauer character.

(f) $\varphi_{U\oplus V} = \varphi_U + \varphi_V$ and $\varphi_{U\otimes_k V} = \varphi_U \cdot \varphi_V\,.$

(g) Assume U is a liftable and $\widehat{U}$ is a lift, i.e. $\widehat{U}/\mathfrak{p}\widehat{U} \cong U$. Write $\chi_{\widehat{U}}$ be the F-character of $F\otimes_{\mathcal{O}}\widehat{U}$. Then $\varphi_U(g) = \chi_{\widehat{U}}(g)$ on all p-regular elements $g \in G$.

Brauer proved that Brauer characters can be counted using conjugacy classes as well:

Theorem 16.5

The set $\mathrm{IBr}_p(G)$ of irreducible Brauer characters of G forms an F-basis of the F-vector space $\mathrm{Cl}_F(G_{p'})$ of class functions on $G_{p'}$ and

$$|\,\mathrm{IBr}_p(G)| = \dim_F \mathrm{Cl}_F(G_{p'}) = \text{number of conjugacy classes of } p\text{-regular elements in } G\,.$$

We note that the second equality is obvious, because the indicator functions on the conjugacy classes of p-regular elements form an F-basis.

17 Back to reduction modulo $\mathfrak{p}$

We now want to investigate the connections between representations of G over F (or $\mathbb{C}$) and representations of G over k through the connections between their F-characters and Brauer characters.

Proposition 17.1

Let V be an FG-module with F-character χ_V. Then:

(a) there exists an $\mathcal{O}G$-lattice L such that $V \cong F\otimes_{\mathcal{O}} L$ (called an **$\mathcal{O}$-form of** V);

(b) $\chi_V|_{G_{p'}} = \varphi_{\overline{L}}$ and is called the **reduction modulo p of** χ_V;

(c) if $V \in \mathrm{Irr}_F(G)$, there exist non-negative integers $d_{\chi\varphi}$ such that

$$\chi_V|_{G_{p'}} = \sum_{\varphi\in\mathrm{IBr}_p(G)} d_{\chi\varphi}\varphi.$$

Exercise 17.2

Assume G is a p-group. Prove that the reduction modulo p of any linear character is the trivial Brauer character.

Definition 17.3

The matrix

- $D := \mathrm{Dec}_p(G) = (d_{\chi\varphi})_{\substack{\chi \in \mathrm{Irr}_F(G)\\ \varphi \in \mathrm{IBr}_p(G)}}$ is the p**-decomposition matrix** of G;
- $C := D^{tr}D = (c_{\varphi\mu})_{\varphi,\mu \in \mathrm{IBr}_p(G)}$ is the **Cartan matrix** of G.

Proposition 17.4

(a) The decomposition matrix $\mathrm{Dec}_p(G)$ has full rank, namely $|\,\mathrm{IBr}_p(G)|$.

(b) The Cartan matrix of G is a symmetric positive definite matrix with non-negative integer entries.

Recall now that projective kG-modules are liftable and this enables us to associate an F-character of G to each PIM of kG, in fact in a unique way in this case.

Definition 17.5

Let $\varphi \in \mathrm{IBr}_p(G)$ be an irreducible Brauer character afforded by a simple kG-module S. Let P_S be the projective cover of S and let $\widehat{P}_S$ denote a lift of P_S to $\mathcal{O}$. Then, the F-character of $(\widehat{P}_S)^F$ is denoted by Φ_φ and is called the **projective indecomposable character** associated to S or φ.

Proposition 17.6

Let $\varphi \in \mathrm{IBr}_p(G)$. Then:

(a) $\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}_F(G)} d_{\chi\varphi}\chi$; and

(b) $\Phi_\varphi|_{G_{p'}} = \sum_{\mu \in \mathrm{IBr}_p(G)} c_{\varphi\mu}\mu$.

Definition 17.7 (*Brauer character table*)

Set $l := |G_{p'}|$ and let $g_1, \ldots, g_l$ be a complete set of representatives of the p-regular conjugacy classes of G.

(a) The **Brauer character table** of a finite group G is the matrix $\Big(\varphi(g_j)\Big)_{\substack{\varphi \in \mathrm{IBr}_p(G)\\ 1 \leqslant j \leqslant l}} \in M_l(F)$.

(b) The **Brauer projective table** of a finite group G at p is the matrix $\Big(\Phi_\varphi(g_j)\Big)_{\substack{\varphi \in \mathrm{IBr}_p(G)\\ 1 \leqslant j \leqslant l}} \in M_l(F)$.

Wednesday. Chapter 5. Block Theory

We now want to break down the representation theory of finite groups into its smallest parts: the *blocks* of the group algebra. Before we proceed, I want to give the following warning: one of the confusing things about the block theory of finite groups is that there often seems to be more than one definition of the same concept. In fact several different definitions – and mathematical objects – are hidden behind the word *block* of a group algebra. Some texts consider blocks to be *algebras*, or more precisely indecomposable 2-sided ideals of the group algebra, some to be *primitive central idempotents* of the group algebra, some to be the union of the sets of irreducible ordinary characters and irreducible Brauer characters of the aforementioned algebra, some others to be an equivalence class of modules over the group algebra (sometimes simple, sometimes indecomposable, sometimes arbitrary),... Important is to keep in mind, that although different authors use different approaches, there are essentially equivalent. We will focus here on the algebra approach.

Notation: We keep the notation and the assumptions of the previous Chapters. Throughout, G denotes a finite group, p a prime number. We let $(F, \mathcal{O}, k)$ denote a p-modular system and we assume F contains all $\exp(G)$-th roots of unity, so $(F, \mathcal{O}, k)$ is a splitting p-modular system for G and all its subgroups (see Theorem 15.2). We write $\mathfrak{p} := J(O)$ and we let $\Lambda \in \{F, \mathcal{O}, k\}$.

18 The p-Blocks of a Group

The *block decomposition* of the group algebra ΛG is just the the decomposition of ΛG, seen a $(\Lambda G, \Lambda G)$-bimodule, into indecomposable $(\Lambda G, \Lambda G)$-bimodules. So in block theory of finite groups, by definition, one should work with bimodules. However, bimodules over group algebras can always be made into one-sided modules as described in the following remark.

Remark 18.1

Let G_1 and G_2 be finite groups. If M is a $(\Lambda G_1, \Lambda G_2)$-bimodule, then M can be endowed with the structure of a one-sided $\Lambda[G_1 \times G_2]$-module via the $G_1 \times G_2$-action:

$$\cdot : (G_1 \times G_2) \times M \longrightarrow M,\ m \mapsto (g, h) \cdot m := g \cdot m \cdot h^{-1}$$

The consequence is that the one-sided module theoretic terms and results we have seen so far can be applied to such bimodules. Thus, in the sequel, we identify bimodules with one-sided left modules without further mention.

Definition 18.2 (***Blocks of the group algebra***)

In the unique decomposition $\Lambda G = B_1 \oplus \cdots \oplus B_n$ into indecomposable $(\Lambda G, \Lambda G)$-subbimodules of ΛG, the summands $B_1, \ldots, B_n$ are called the **blocks** of ΛG. (Or sometimes just **block algebras**.)

Remark 18.3

The block decomposition $\Lambda G = B_1 \oplus \cdots \oplus B_n$ is equivalent to a decomposition

$$1 = \underbrace{e_1}_{\in B_1} + \ldots + \underbrace{e_n}_{\in B_n}$$

of the unit of ΛG as a sum of orthogonal primitive central idempotents $e_i \in Z(\Lambda G)$, where $e_i = 1_{B_i}$ and $B_i = \Lambda G e_i \, \forall \, 1 \leqslant i \leqslant n$. We call the elements $e_1, \ldots, e_n$ the **block idempotents** of ΛG.

Definition 18.4 (***belonging to a block***)

We say that an (indecomposable) ΛG-module M **belongs to (or lies in) the block** $B_i = \Lambda G e_i$ if $e_i M = M$ and $e_j M = 0$ for all $1 \leqslant j \leqslant n$ such that $j \neq i$.

Remark 18.5

It follows from the previous remark, that every *indecomposable* ΛG-module M belongs to a uniquely determined block of ΛG. Indeed, the decomposition

$$1 = e_1 + \ldots + e_n \implies M = 1 \cdot M = e_1 \cdot M \oplus \ldots \oplus e_n \cdot M$$

but as M is indecomposable the Krull-Schmidt theorem tells us that

$$\exists! \, 1 \leqslant i \leqslant n \text{ such that } e_i M = M \text{ and } e_j M = 0 \; \forall \, 1 \leqslant j \leqslant n \text{ with } j \neq i \, .$$

Definition 18.6 (***Principal block***)

The principal block of ΛG is the block to which the trivial module Λ belongs. **Notation**: $B_0(\Lambda G)$.

Exercise 18.7

(a) Let B_i be a block of ΛG and let e_i be the corresponding block idempotent. Prove that a ΛG-module M belongs to B_i if and only if external multiplication by e_i is a ΛG-isomorphism on that module.

(b) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of ΛG-modules and ΛG-homomorphisms. Prove that, for each $1 \leqslant i \leqslant n$:

$$M \text{ belong to the block } B_i \text{ of } \Lambda G \text{ if and only } L \text{ and } N \text{ belong to } B_i \, .$$

[Hint: use (a) and the 5-Lemma.]

(c) Deduce that if a ΛG-module M lies in a block B of ΛG, then so do all of its submodules and all of its factor modules.

Example 15 (***Blocks of FG***)

Since FG is semisimple, the block decomposition of FG is given by the Artin-Wedderburn Theorem. In particular, the blocks are matrix algebras and can be labelled by $\mathrm{Irr}(FG)$. (Or $\mathrm{Irr}_F(G)$ if you prefer!)

Remark 18.8 (***Blocks of $\mathcal{O}G$ and kG***)

The Lifting of Idempotents tells us that the quotient morphism $\mathcal{O}G \twoheadrightarrow [\mathcal{O}/\mathfrak{p}]G = kG, x \mapsto \overline{x}$ induces a bijection

$$\begin{array}{rcl} \{\text{primitive idempotents of } Z(\mathcal{O}G)\} & \xleftrightarrow{\sim} & \{\text{primitive idempotents of } Z(kG)\} \\ e & \mapsto & \overline{e}\ . \end{array}$$

Thus a decomposition $1_{\mathcal{O}G} = e_1 + \cdots + e_r$ of the identity element of $\mathcal{O}G$ into a sum of primitive central idempotents corresponds to a decomposition $1_{kG} = \overline{e}_1 + \cdots + \overline{e}_r$ of the identity element of kG into a sum of primitive central idempotents of kG. Therefore, by Proposition 18.3, there is a bijection between the blocks of $\mathcal{O}G$ and the blocks of kG:

$$\begin{array}{ccccccccc} \mathcal{O}G & = & B_1 & \oplus & \cdots & \oplus & B_n \\ \big\downarrow & & \updownarrow & & \cdots & & \updownarrow \\ kG & = & \overline{B_1} & \oplus & \cdots & \oplus & \overline{B_n} \end{array}$$

We define a p**-block** of G to be the specification of a block of $\mathcal{O}G$, understanding also the corresponding block of kG. We write $\mathrm{Bl}_p(G)$ for the set of all p-blocks of G when it is clear from the context/unimportant whether we work over $\mathcal{O}$ or over k, resp. $\mathrm{Bl}_p(\mathcal{O}G)$ for the set of all blocks of $\mathcal{O}G$ and $\mathrm{Bl}_p(kG)$ for the set of all blocks of kG.

The division of the simple kG-modules into blocks can be achieved in a purely combinatorial fashion, knowing the Cartan matrix of kG. The connection with a block matrix decomposition of the Cartan matrix is probably the origin of the use of the term *block* in representation theory.

Remark 18.9

On listing the simple kG-modules so that modules in each block occur together, the Cartan matrix of kG has a block diagonal form, with one block matrix for each p-block of the group. Up to permutation of simple modules within p-blocks and permutation of the p-blocks, this is the unique decomposition of the Cartan matrix into block diagonal form with the maximum number of block matrices.

19 Defect Groups

From now on we will only discuss the blocks of kG. (Analogous results hold for the corresponding blocks of $\mathcal{O}G$.) We write $\Delta : G \longrightarrow G \times G, g \mapsto (g, g)$ for the diagonal embedding of G in $G \times G$.

We start with a result, which lets us identify the vertices of a block with a conjugacy class of p-subgroups of G.

Theorem 19.1

If $B \in \mathrm{Bl}_p(kG)$, then every vertex of B, considered as an indecomposable $k[G \times G]$-module, has the form $\Delta(D)$ for some p-subgroup $D \leqslant G$. Moreover, D is uniquely determined up to conjugation in G.

Definition 19.2 (*Defect group, defect*)

Let $B \in \mathrm{Bl}_p(kG)$.

(a) A **defect group** of B is a p-subgroup $D \leqslant G$ such that $\Delta(D)$ is a vertex of B considered as an indecomposable $k[G \times G]$-module.

(b) If $|D| = p^d$ ($d \in \mathbb{Z}_{\geqslant 0}$) then d is called the **defect** of B.

Note. As the vertices of a module form a a conjugacy class of subgroups, so do the defect groups of a block and it is clear that in fact all defect groups have the same order.

Defect groups are useful and important because in some sense they measure how far a p-block is from being semisimple (see Exercise 19.5 below). In general they are very difficult to determine concretely. However, the following properties (mostly due to Green) are useful.

Properties 19.3

Let $B \in \mathrm{Bl}_p(kG)$ with defect group $D \leqslant G$. Then the following assertions hold.

(a) If B is a block of kG with defect group D, then every indecomposable kG-module belonging to B is relatively D-projective, and hence has a vertex contained in D.

(b) D contains every normal p-subgroup of G;

(c) D is the largest normal p-subgroup of $N_G(D)$, i.e. $Q = O_p(N_G(Q))$.

Example 16

Since the vertices of the trivial kG-module k are the Sylow p-subgroups of G, so are the defect groups of the principal group $B_0(kG)$.

Exercise 19.4 (*p-block(s) of a p-group*)

Prove that if G is a p-group, then G has a unique p-block.

Exercise 19.5

Let B be a block of kG with a trivial defect group. Prove that B is a semisimple algebra.

Finally we present a fundamental result due to Brauer.

Definition 19.6

Let $H \leqslant G$, let $b \in \mathrm{Bl}_p(kH)$. Then a block $B \in \mathrm{Bl}_p(kG)$ **corresponds to** b if and only if $b \mid B{\downarrow}^{G\times G}_{H\times H}$ *and* B is the unique block of kG with this property. We then write $B = b^G$. If such a block B exists, then we say that b^G **is defined**.

Theorem 19.7 (*Brauer's First Main Theorem*)

Let $D \leqslant G$ be a p-subgroup and let $H \leqslant G$ containing $N_G(D)$. Then, there is a bijection

$$\{\text{Blocks of } kH \text{ with defect group } D\} \xrightarrow{\sim} \{\text{Blocks of } kG \text{ with defect group } D\}$$
$$b \mapsto b^G$$

Moreover, in this case b^G is called the **Brauer correspondent** of b (and conversely).

Proof (Sketch): This is a particular case of the Green correspondence (i.e. when viewing blocks as one-sided left modules). ■

Many of the results and open problems in modular representation theory of finite groups are concerned with the influence of the structure of the defect group on the structure of the block. For example, by a result of Brauer, $|D|$ is the largest elementary divisor of the Cartan matrix of a block B with defect group D, and it appears with multiplicity 1. We mention here two major open problems in this spirit.

Conjecture 19.8 (*Brauer's* $\mathtt{k}(B)$*-Conjecture*)

Let $B \in \mathrm{Bl}_p(kG)$ with defect group D. Then $|\operatorname{Irr}_F(B)| \leqslant |D|$.

Conjecture 19.9 (*Broué's Abelian Defect Group Conjecture*)

Let $B \in \mathrm{Bl}_p(G)$ with abelian defect group D and let $b \in \mathrm{Bl}_p(N_G(D))$ be the Brauer correspondent of B. Then, the derived categories $D^b(\mathrm{mod}(B))$ and $D^b(\mathrm{mod}(b))$ of bounded complexes of finitely generated modules over B and b are equivalent as triangulated categories.

20 Equivalences of Block Algebras

Basic Question 20.1 (*Open!!*)

Which k-algebras (resp. $\mathcal{O}$-algebras) occur as p-blocks of finite groups?

Conjectural Answer 20.2

If a defect group is fixed, only finitely many ... up to a *good* notion of equivalence!

In this respect, Donovan's and Puig's Conjectures are further good examples of open problems concerned with the influence of the structure of the defect group on the structure of the block.

Conjecture 20.3 (***Donovan's/Puig's Conjecture, '70's/'80's***)

Let D be a finite p-group. Then, there exists only finitely many (splendid) Morita equivalence classes of p-blocks of finite groups with a defect group isomorphic to D.

Donovan's Conjuecture is known to hold over $\mathcal{O}$ and over k for a fairly long list of *small* defect groups. The status of this conjecture is kept up-to-date by Charles Eaton on the Wiki page of his *block library*. See `https://wiki.manchester.ac.uk/blocks/index.php/Main_Page`.

On the other hand, not much is known towards Puig's Conjecture. It is known to hold if $D \cong C_{p^n}$, that is, is a cyclic p-group (Linckelmann, 1996) and if $D \cong C_2 \times C_2$ if $p = 2$ (Craven-Eaton-Kessar-Linckelmann, 2012).

Here:

Definition 20.4 (***Morita equivalence***)

Let G and G' be two finite groups. Two block algebras $A \in \mathrm{Bl}_p(G)$ and $B \in \mathrm{Bl}_p(G')$ are called **Morita equivalent** iff $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ are equivalent as (k-linear, resp. $\mathcal{O}$-linear) categories. If this is the case, then we write $A \sim_M B$.

The following result on Morita equivalences is often useful in order to verify that such an equivalence exists.

Theorem 20.5 (***Morita's Theorem***)

With the assumptions and notation of the previous definition, TFAE:

(a) $A \sim_M B$; and

(b) there exists an (A, B)-bimodule M and a (B, A)-bimodule N such that $M \otimes_B N \cong A$ (as (A, A)-bimodules) and $N \otimes_A M \cong B$ (as (B, B)-bimodules).

In fact in the case of block algebras, N is the dual of M. Therefore, we often say that the Morita equivalence is **induced** of **realised** by the bimodule M of Assertion (b) of Morita's Theorem.

Definition 20.6 (***Morita equivalence***)

Let G and G' be two finite groups. Assume M is an (A, B)-bimodule realising a Morita equivalence bewtween $A \in \mathrm{Bl}_p(kG)$ and $B \in \mathrm{Bl}_p(kG')$. This Morita equivalence is called:

- a **splendid Morita equivalence** (or also a **source-algebra equivalence** or a **Puig equivalence**) iff the bimodule M, seen as a left $k[G \times G']$-module, is a p-permutation module, and if it is the case we write $A \sim_{SM} B$
- an **endo-permutation source equivalence** (or also a **basic equivalence**) iff the bimodule M, seen as a left $k[G \times G']$-module, has a source T such that $\mathrm{End}_k(T) \cong$ permutation module.

Morita and splendid Morita equivalences of occur naturally in the modular representation theory of finite groups. Standard examples are as follows:

Example 17 (***Examples of (splendid) Morita equivalences in the block theory of finite groups***)

(a) Isomorphic blocks (i.e. as k-algebras) are always Morita equivalent.

(b) $B_0(kG) \sim_{SM} B_0(k[G/O_{p'}(G)])$ because $O_{p'}(G)$ always acts trivially on the principal block.

(c) **"Alperin/Dade"**. If $G \trianglelefteq \widetilde{G}$ and there is a Sylow p-subgroup P of G such that $\widetilde{G} = GC_{\widetilde{G}}(P)$, then

$$B_0(k\widetilde{G}) \sim_{SM} B_0(kG)\,.$$

In fact in this case the two principal blocks are isomorphic.

(d) **"Fong-Reynolds"**. If $H \trianglelefteq G$, $b \in \mathrm{Bl}_p(H)$, $T := \mathrm{Stab}_G(b)$, then there exists a bijection

$$\mathrm{Bl}_p(T \mid b) \xrightarrow{\sim} \mathrm{Bl}_p(B \mid b), B \mapsto B^G$$

where the bimodule $M := 1_{B^G} \cdot kG \cdot 1_B$ realises a splendid Morita equivalence between B and B^G.

Remark 20.7

It can be proved that splendidly Morita equivalent blocks and basically equivalent blocks necessarily have isomorphic defect groups.
Whether Morita equivalent blocks necessarily have isomorphic defect groups was an open question for a long time. However, as mentioned by Claudio in his talk, a special case is the *modular isomorphism problem*, which has recently (July 2021) been shown to have a negative answer by Garcia-Margolis-Del Rio. More precisely, they prove that there are non-isomorphic finite 2-groups G and G' such that the group rings of G and G' over any field of characteristic 2 are isomorphic.

Finally we mention that the notions of a Morita and a splendid Morita equivalence can be weakened in different flavours to equivalences between the stable module categories or of the bounded derived categories of the blocks.

Definition 20.8 (***Rickard equivalence / Stable equivalence of Morita type***)

Let G and G' be two finite groups. Two block algebras $A \in \mathrm{Bl}_p(G)$ and $B \in \mathrm{Bl}_p(G')$ are called:

(a) **Rickard** (or **derived**) **equivalent** if the derived categories $D^b(\mathrm{mod}(A))$ and $D^b(\mathrm{mod}(B))$ of bounded complexes of finitely generated modules over A and B are equivalent as triangulated categories.

(b) "**stably equivalent à la Morita**" (or say that there is a **stable equivalence of Morita type** between A and B) if there exist an (A, B)-bimodule M which is projective as an A-module and as a B-module and a (B, A)-bimodule N which is projective as a B-module and as an A-module such that $M \otimes_B N \cong A \oplus$ (projectives) as (A, A)-bimodules and $N \otimes_A M \cong B \oplus$ (projectives) as (B, B)-bimodules.

Remark 20.9

(a) A derived version of Morita's theorem asserts that A and B are Rickard equivalent if and only if the equivalence of triangulated categories between $D^b(\mathrm{mod}(A))$ and $D^b(\mathrm{mod}(B))$ can be realised by tensoring over B with a bounded complex $M_\bullet$ of (A, B)-bimodules in which each term is both projective as an A-module and as B-module. When all terms in this complex (seen as one-sided left modules) are p-permutation modules, then the equivalence is called a **splendid Rickard equivalence**.

(b) A stable equivalence of Morita type between A and B induces an equivalence of triangulated categories between the stable categories $\mathrm{stmod}(A)$ and $\mathrm{stmod}(B)$.

See the HANDOUT of my Beamer presentation for relations between these equivalences.

Finally, we mention that blocks with cyclic defect groups are a very nice playground to play around with several notions of equivalences – mentioned in this section – and more concepts such as the Green correspondence, Clifford theory or perfect isometries of characters.

Remark 20.10 (*Blocks with cyclic defect groups*)

Let $B \in \mathrm{Bl}_p(kG)$ be a block with a cyclic defect group D. Let D_1 be the unique cyclic subgroup of D of order p. As D is cyclic, $N_1 := N_G(D_1) \geqslant N_G(D)$, so we may consider the Brauer correspondent $b \in \mathrm{Bl}_p(N_G(D_1))$ of B, let $c \in \mathrm{Bl}_p(C_G(D_1))$ be a block of $C_G(D_1)$ covered by b and let $b' \in \mathrm{Bl}_p(\mathrm{Stab}_{N_G(D_1)}(c))$ be the Fong-Reynolds correspondent of c. Then, we have the following situation:

$$\begin{tikzcd}
G \arrow[d, dash] & B-\mathrm{mod} \arrow[d, shift left, "\substack{\text{stable equivalence of Morita type, induced by the Green correspondence}\\ \text{(+ a perfect isometry)}}"] \\
N_1 \arrow[d, dash] & b-\mathrm{mod} \arrow[u, shift left] \arrow[d, shift left, "\text{splendid Morita equivalence (given by the Fong-Reynolds correspondence, induced by } \mathrm{Ind}_T^{N_1})"] \\
\mathrm{Stab}_{N_1}(c) \arrow[d, dash] & b'-\mathrm{mod} \arrow[u, shift left] \\
C_G(D_1) & c-\mathrm{mod} \arrow[u, dashed, "\text{Clifford theory (induced by } \mathrm{Ind}^T_{C_G(D_1)})"']
\end{tikzcd}$$