## Chapter 9. Lifting Results and Brauer's Reciprocity

In this chapter, we consider lifting results from positive characteristic to characteristic zero. In particular, we prove that projective $kG$-modules, and more generally $p$-permutation $kG$-modules are liftable modules. We use $p$-modular systems $(F, \mathcal{O}, k)$ and *Brauer 's reciprocity theorem* to gain information about $kG$ and its modules (which is/are extremely complicated) from the group algebra $FG$, which is semisimple and therefore much better understood, via the group algebra $\mathcal{O}G$.

**Notation**. Throughout this chapter, unless otherwise specified, we assume that Assumption $(*)$ holds.

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## 32 Lifting idempotents and projective modules

Throughout this section, we may simply assume that $\mathcal{O}$ is a complete discrete valuation ring with unique maximal ideal $\mathfrak{p} := J(\mathcal{O})$ and residue field $k := \mathcal{O}/\mathfrak{p}$. In particular, the case $\mathcal{O} = k$ is allowed. Moreover, we let $A$ denote an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module. Observe that since $A$ is a finitely generated $\mathcal{O}$-module, so is any simple $A$-module $V$ and it follows from Nakayama's lemma that $\mathfrak{p}V \neq V$, so $\mathfrak{p}V = 0$. If $\mathfrak{m}$ is a maximal left ideal of $A$, then $A/\mathfrak{m}$ is a simple $A$-module and therefore $\mathfrak{m} \supseteq \mathfrak{p}A$. This proves that $\mathfrak{p}A \subseteq J(A)$ . It follows that $J(A)$ is the inverse image in $A$ under the quotient morphism of the Jacobson radical $J(\overline{A})$ of the finite dimensional $k$-algebra $\overline{A} := A/\mathfrak{p}A$ (the reduction modulo $\mathfrak{p}$ of $A$). Consequently,

$$A/J(A) \cong \overline{A}/J(\overline{A}).$$

Recall that an idempotent element $e$ of a ring $R$ is called primitive if $e \neq 0$ and whenever $e = f + g$ where $f$ and $g$ are orthogonal idempotents, then either $f = 0$ or $g = 0$.

**Exercise 32.1**

Let $A$ be a finitely generated $\mathcal{O}$-algebra and let $e \in A$ be an idempotent element. Prove that:

(a) $J(eAe) = eJ(A)e$;

(b) if $M$ is an $A$-module, then $\mathrm{Hom}_A(Ae, M) \cong eM$ as $\mathrm{End}_A(M)$-modules;

(c) $e$ is primitive if and only if the left ideal $Ae$ is indecomposable if and only if $e$ and $0$ are the only idempotents of $eAe$;

(d) if $A$ is a finite-dimensional $k$-algebra, then $e$ is primitive if and only if $eAe$ is a local ring (in which case $eJ(A)e$ is the unique maximal ideal of $eAe$).

This leads us to the following crucial result for representation theory of finite groups on the lifting of idempotents.

**Theorem 32.2 (*Lifting theorem of idempotents (partial version)*)**

Let $A$ be a finitely generated $\mathcal{O}$-algebra. Set $\overline{\overline{A}} := A/J(A)$ and for $a \in A$ write $\overline{\overline{a}} := a + J(A)$. The following assertions hold.

(a) If $\overline{\overline{a}} \in \overline{\overline{A}}^{\times}$, then $a \in A^{\times}$. Thus there is a s.e.s. of groups

$$1 \longrightarrow 1 + J(A) \longrightarrow A^{\times} \longrightarrow \overline{\overline{A}}^{\times} \longrightarrow 1.$$

(b) For any idempotent $x \in \overline{\overline{A}}$, there exists an idempotent $e \in A$ such that $\overline{\overline{e}} = x$.

(c) Two idempotents $e, f \in A$ are conjugate in $A$ if and only if $\overline{\overline{e}}$ and $\overline{\overline{f}}$ are conjugate in $\overline{\overline{A}}$. More precisely if $\overline{\overline{e}} = \overline{\overline{u}}\,\overline{\overline{f}}\,\overline{\overline{u}}^{-1}$, then $\overline{\overline{u}}$ lifts to an invertible element $u \in A^{\times}$ such that $e = ufu^{-1}$. In particular, if $\overline{\overline{e}} = \overline{\overline{f}}$, then there exists $u \in (1 + J(A)) \subseteq A^{\times}$ such that $e = ufu^{-1}$.

(d) An idempotent $e \in A$ is primitive in $A$ if and only if $\overline{\overline{e}}$ is primitive in $\overline{\overline{A}}$.

(e) The quotient morphism $A \to \overline{\overline{A}}$ induces a bijection between the set $\mathcal{P}(A)$ of conjugacy classes of primitive idempotents of $A$ and the set $\mathcal{P}(\overline{\overline{A}})$ of conjugacy classes of primitive idempotents of $\overline{\overline{A}}$.

**Proof:** (a) If $a \in A$ is not invertible, then $Aa \neq A$. Then $a \in \mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of $A$ by Zorn's lemma. Since $\mathfrak{m} \supseteq J(A)$, its image is a maximal left ideal of $\overline{\overline{A}}$ and we have $\overline{\overline{a}} \in \mathfrak{m}$. Thus $\overline{\overline{a}}$

is not invertible. The claim about the s.e.s. follows immediately: $1 + J(A)$ is clearly the kernel of the quotient map.

(b) Let $a_1 \in A$ such that $\overline{\overline{a}}_1 = x$ and let $b_1 := a_1^2 - a_1$. Define recursively two sequences of elements of $A$:

$$a_n := a_{n-1} + b_{n-1} - 2a_{n-1}b_{n-1} \qquad \text{and} \qquad b_n := a_n^2 - a_n \qquad \forall\, n \geqslant 2.$$

We now prove by induction that $a_n^2 \equiv a_n \pmod{J(A)^n}$, or in other words that $b_n \in J(A)^n$. This is true for $n = 1$, and assuming that this holds for $n$, we have $b_n^2 \in J(A)^{n+1}$ (because $(J(A)^n)^2 \subseteq J(A)^{n+1}$), and since $a_n^2 = a_n + b_n$ we obtain

$$\begin{aligned} a_{n+1}^2 &\equiv a_n^2 + 2a_nb_n - 4a_n^2b_n \pmod{J(A)^{n+1}} \\ &= a_n + b_n + 2a_nb_n - 4(a_n + b_n)b_n \\ &\equiv a_n + b_n - 2a_nb_n = a_{n+1} \pmod{J(A)^{n+1}}. \end{aligned}$$

It follows that $(b_n)_{n\geqslant 1}$ converges to $0$ and that $(a_n)_{n\geqslant 1}$ is a Cauchy sequence in the $J(A)$-adic topology. Since $A$ is complete in the $J(A)$-adic topology (see Remark 13.7), $(a_n)_{n\geqslant 1}$ converges to some element $\tilde{e} \in A$. Clearly, $\tilde{e}^2 - \tilde{e} = \lim_{n\to\infty} b_n = 0$, so $\tilde{e}$ is an idempotent of $A$. Moreover

$$\overline{\overline{\tilde{e}}} = \overline{\overline{a}}_1 = x\,,$$

as required.

(c) It is clear that $\overline{\overline{e}}$ and $\overline{\overline{f}}$ are conjugate in $\overline{\overline{A}}$ if $e$ and $f$ are conjugate in $A$ ($e = ufu^{-1} \Rightarrow \overline{\overline{e}} = \overline{\overline{u}}\,\overline{\overline{f}}\,\overline{\overline{u}}^{-1}$). Conversely, assume that there exists $\overline{\overline{u}} \in \overline{\overline{A}}^{\times}$ such that $\overline{\overline{e}} = \overline{\overline{u}}\,\overline{\overline{f}}\,\overline{\overline{u}}^{-1}$. Then, by (a), $u \in A^{\times}$ and so, replacing $f$ by $ufu^{-1}$ we can assume $\overline{\overline{e}} = \overline{\overline{f}}$. Now let $v := 1_A - e - f + 2ef$. Then, by (a), $v \in A^{\times}$ because

$$\overline{\overline{v}} = \overline{\overline{1_A - e - f + 2ef}} = \overline{\overline{1}}_A - \overline{\overline{e}} - \overline{\overline{f}} + 2\overline{\overline{e}}\,\overline{\overline{f}} = 1_{\overline{\overline{A}}}\,.$$

Moreover, $ev = ef = vf$ and it follows that $e = evv^{-1} = vfv^{-1}$, as required.

(d) By the previous exercise the idempotent $e \in A$ is primitive if and only if $e$ and $0$ are the only idempotents of $eAe$ and $J(eAe) = eJ(A)e = J(A) \cap eAe$. Thus,

$$eAe/J(eAe) = \overline{\overline{eAe}} = \overline{\overline{e}}\,\overline{\overline{A}}\,\overline{\overline{e}}\,.$$

Now, if $\overline{\overline{f}}$ is a non-trivial idempotent of $\overline{\overline{e}}\,\overline{\overline{A}}\,\overline{\overline{e}}$, then by (b) applied to the algebra $eAe$, the idempotent $\overline{\overline{f}}$ lifts to an idempotent $f \in eAe$. This proves that $\overline{\overline{e}}$ is primitive if $e$ is primitive. Conversely if $e$ is not primitive, there exists a non-trivial idempotent $f \in eAe$. Then $f$ is not conjugate (that is, not equal to $0$ nor to the unity element $e$). Thus, it follows from (c) that $\overline{\overline{f}}$ is a non-trivial idempotent of $\overline{\overline{e}}\,\overline{\overline{A}}\,\overline{\overline{e}}$, as required.

(e) This follows immediately from (b), (c) and (d). ■

A first possible application of Theorem 32.2 is a generalisation of this theorem providing a lifting of idempotents from a quotient $A/\mathfrak{b}$ for an arbitrary ideal $\mathfrak{b} \in A$. (See [Thé95, (3.2) Theorem].) We state here the particular case of interest to us for $\mathfrak{b} = \mathfrak{p}A$.

**Theorem 32.3 (*Lifting theorem of idempotents for reduction modulo* $\mathfrak{p}$)**

Let $A$ be a finitely generated $\mathcal{O}$-algebra. Set $\overline{A} := A/\mathfrak{p}A$ and for $a \in A$ write $\overline{a} := a + \mathfrak{p}A$. The following assertions hold.

(a) For every idempotent $x \in \overline{A}$, there exists an idempotent $e \in A$ such that $\overline{e} = x$.

(b) The units of $A$ are $A^{\times} = \{a \in A \mid \overline{a} \in \overline{A}^{\times}\}$.

(c) If $e_1, e_2 \in A$ are idempotents such that $\overline{e}_1 = \overline{e}_2$ then there is a unit $u \in A^{\times}$ such that $e_1 = ue_2u^{-1}$ .

(d) The quotient morphism $A \twoheadrightarrow \overline{A}$ induces a bijection between the central idempotents of $A$ and the central idempotents of $\overline{A}$.

**Proof:** Exercise! (Either use Theorem 32.2 or imitate the proof of Theorem 32.2.) ■

The lifting of idempotents allows us, in particular, to prove that projective indecomposable $kG$-modules are liftable to projective indecomposable $\mathcal{O}G$-lattices.

**Lemma 32.4**

Let $A$ be a finitely generated algebra over a commutative ring $R$. If $P$ is a projective indecomposable $A$-module, then there exists an idempotent $e \in A$ such that $P \cong Ae$.

**Proof:** Since $P$ is projective, $P \mid (A^{\circ})^n$ for some $n \in \mathbb{Z}_{\geqslant 1}$. As $P$ is indecomposable, it follows from the Krull-Schmidt theorem that $P \mid A^{\circ}$, so $A^{\circ} = P \oplus Q$ for some $A$-module $Q$. Thus, we can write $1_A = e + f$ with $e \in P$ and $f \in Q$. Then

$$fe = (1-e)e = e - e^2 = ef$$

and this is an element of $Ae \cap Af \subseteq Q \cap P = \{0\}$. Therefore $e^2 = e$ and $ef = fe = 0$. Finally, since

$$A = A \cdot 1_A = A(e+f) = Ae + Af\,, \;\; Ae \subseteq P \;\text{ and }\; Af \subseteq Q\,,$$

it follows that $Ae = P$. ■

**Corollary 32.5**

Let $A$ be a finitely generated $\mathcal{O}$-algebra and set $\overline{A} := A/\mathfrak{p}A$. Let $P$ be a projective (indecomposable) $\overline{A}$-module. Then there exists a projective (indecomposable) $A$-module $\widehat{P}$ such that $P \cong \widehat{P}/\mathfrak{p}\widehat{P}$.

**Proof:** Let $P = P_1 \oplus \cdots \oplus P_r$ ($r \in \mathbb{Z}_{\geqslant 1}$) be a decomposition of $P$ as a direct sum of indecomposable $A$-submodules. Then, by Lemma 32.4, there exist idempotents $f_1, \ldots, f_r \in \overline{A}$ such that $P_i \cong \overline{A}f_i$ for each $1 \leqslant i \leqslant r$, and so

$$P \cong \overline{A}f_1 \oplus \cdots \oplus \overline{A}f_r\,.$$

Now, by Theorem 32.3(a) there exists idempotents $e_1, \ldots, e_r \in A$ such that $\overline{e_i} = f_i$ for each $1 \leqslant i \leqslant r$. Then $\widehat{P} := Ae_1 \oplus \cdots \oplus Ae_r$ is a projective $A$-module (see Example 13) and $\widehat{P}/\mathfrak{p}\widehat{P} \cong P$.
Moreover, $\widehat{P}$ is indecomposable if $P$ is. Indeed,

$$\widehat{P} = \widehat{P}_1 \oplus \widehat{P}_2 \text{ decomposable } \Rightarrow \widehat{P}/\mathfrak{p}\widehat{P} \cong k \otimes_{\mathcal{O}} (\widehat{P}_1 \oplus \widehat{P}_2) \cong (k \otimes_{\mathcal{O}} \widehat{P}_1) \oplus (k \otimes_{\mathcal{O}} \widehat{P}_2) \text{ decomposable}\,.$$

■

**Corollary 32.6**

Any (projective) indecomposable $kG$-module is liftable to a (projective) indecomposable $\mathcal{O}G$-lattice.

**Proof:** This follows immediately from Corollary 32.5 with $A := \mathcal{O}G$ since then

$$\overline{A} = \mathcal{O}G/\mathfrak{p}\mathcal{O}G \cong k \otimes_{\mathcal{O}} \mathcal{O}G \cong (k \otimes_{\mathcal{O}} \mathcal{O})G \cong kG\,.$$

■

# 33 Lifting $p$-permutation modules

We now prove that the lifting of idempotents can also be used to prove that any $p$-permutation $kG$-module is liftable to an $\mathcal{O}G$-lattice. This due to Scott, who in fact proved that the $k$-endomorphism ring of a transitive permutation $kG$-module is liftable, thus generalising the previous result on projective modules. However, we first want to see that reduction modulo $\mathfrak{p}$ preserves the property of being a $p$-permutation module.

**Exercise 33.1**

Prove that if $L$ is an indecomposable $p$-permutation $\mathcal{O}G$-lattice with vertex $Q$, then $L/\mathfrak{p}L$ is an indecomposable $p$-permutation $kG$-module with vertex $Q$.

**Exercise 33.2**

Assume $K \in \{\mathcal{O}, k\}$. Recall that a *primitive decomposition* of an idempotent element $e \in KG$ is a decomposition of $e$ of the form $e = \sum_{i \in I} i$ where $I$ is a set of pairwise orthogonal primitive idempotents of $M$. Prove that a decomposition of a $KG$-module $M$ into a direct sum of indecomposable summands amounts to choosing a primitive decomposition of $\mathrm{Id}_M \in \mathrm{End}_{KG}(M)$.

**Theorem 33.3 (*L. Scott, 1973*)**

(a) If $L_1$ and $L_2$ are $p$-permutation $\mathcal{O}G$-lattices, then the natural homomorphism

$$\mathrm{Hom}_{\mathcal{O}G}(L_1, L_2) \longrightarrow \mathrm{Hom}_{kG}(L_1/\mathfrak{p}L_1, L_2/\mathfrak{p}L_2),\ \varphi \mapsto \bar{\varphi}$$

induced by reduction modulo $\mathfrak{p}$ is surjective.

(b) Every $p$-permutation $kG$-module lifts to a $p$-permutation $\mathcal{O}G$-lattice.

**Proof:** (a) By the characterisation of $p$-permutation $kG$-modules in Proposition-Definition 30.5 it is enough to prove that assertion (a) holds for transitive permutation $\mathcal{O}G$-lattices. Thus, we may assume $L_1 = \mathrm{Ind}_{Q_1}^G(\mathcal{O})$ and $L_2 = \mathrm{Ind}_{Q_2}^G(\mathcal{O})$ for some $p$-subgroups $Q_1, Q_2 \leqslant G$. Applying Frobenius' reciprocity twice and Mackey's formula we obtain

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}G}(L_1, L_2) &\cong \mathrm{Hom}_{\mathcal{O}G}(\mathrm{Ind}_{Q_1}^G(\mathcal{O}), \mathrm{Ind}_{Q_2}^G(\mathcal{O})) \\
&\cong \mathrm{Hom}_{\mathcal{O}Q_2}(\mathrm{Res}_{Q_2}^G \mathrm{Ind}_{Q_1}^G(\mathcal{O}), \mathcal{O}) \\
&\cong \bigoplus_{x \in [Q_2 \backslash G / Q_1]} \mathrm{Hom}_{\mathcal{O}Q_2}(\mathrm{Ind}_{Q_2 \cap {}^x Q_1}^{Q_2}(\mathcal{O}), \mathcal{O}) \\
&\cong \bigoplus_{x \in [Q_2 \backslash G / Q_1]} \mathrm{Hom}_{\mathcal{O}(Q_2 \cap {}^x Q_1)}(\mathcal{O}, \mathcal{O}) \\
&\cong \bigoplus_{x \in [Q_2 \backslash G / Q_1]} \mathcal{O}\,.
\end{aligned}$$

Hence the $\mathcal{O}$-rank of $\mathrm{Hom}_{\mathcal{O}G}(L_1, L_2)$ is $|Q_2 \backslash G / Q_1|$. The same argument with $k$ instead of $\mathcal{O}$, shows that the $k$-dimension of $\mathrm{Hom}_{kG}(L_1/\mathfrak{p}L_1, L_2/\mathfrak{p}L_2)$ is also $|Q_2 \backslash G / Q_1|$ and surjectivity follows.

(b) Again by the characterisation of $p$-permutation modules in Proposition-Definition 30.5 it suffices to prove that the claim holds for the indecomposable $p$-permutation $kG$-modules. So, let $M$ be an indecomposable $kG$-module with vertex $Q$ and trivial source. Then, $M \mid \mathrm{Ind}_Q^G(k)$ and by Exercise 33.2

there exists a primitive idempotent $\iota \in \mathrm{End}_{kG}(\mathrm{Ind}_Q^G(k))$ such that $M = \iota(\mathrm{Ind}_Q^G(k))$. Now, by (a) the canonical map

$$\mathrm{End}_{\mathcal{O}G}(\mathrm{Ind}_Q^G(\mathcal{O})) \twoheadrightarrow \mathrm{End}_{kG}(\mathrm{Ind}_Q^G(k))$$

is surjective. Therefore, by the lifting theorem of idempotents for reduction modulo $\mathfrak{p}$, there exists an idempotent $\pi \in \mathrm{End}_{\mathcal{O}G}(\mathrm{Ind}_Q^G(\mathcal{O}))$ such that $\iota$ is the reduction modulo $\mathfrak{p}$ of $\pi$. Then $L := \pi(\mathrm{Ind}_Q^G(\mathcal{O}))$ is a direct summand of $\mathrm{Ind}_Q^G(\mathcal{O})$ such that $L/\mathfrak{p}L \cong M$, which is indecomposable and with vertex $Q$ by Exercise 33.1. ■

**Remark 33.4**

In general a $p$-permutation $kG$-module has several lifts. However, amongst all possible lifts there is a unique one, up to isomorphism, which is a $p$-permutation $\mathcal{O}G$-lattice. The fact that the lift produced by the proof of Theorem 33.3(b) is unique up to isomorphism, follows from the fact that primitive decompositions of idempotents are unique up to conjugation.

This leads to the following character-theoretic characterisations of $p$-permutations modules.

**Lemma 33.5**

Let $\widehat{M}$ be an indecomposable $p$-permutation $\mathcal{O}G$-lattice, let $M := \widehat{M}/\mathfrak{p}\widehat{M}$ be its reduction modulo $\mathfrak{p}$, and let $\chi_{\widehat{M}}$ be the character afforded by $F \otimes_{\mathcal{O}} \widehat{M}$. Then the following assertions hold.

(a) If $Q \leqslant G$ is a $p$-subgroup, then $\dim_k(\mathrm{soc}(\mathrm{Res}_Q^G(M))) = \langle \chi_{\widehat{M}}, 1_Q \rangle_Q$.

(b) If $x \in G$ is a $p$-element, then $\chi_{\widehat{M}}(x)$ is equal to the multiplicity of the trivial module as a direct summand of $\mathrm{Res}_{\langle x \rangle}^G(M)$. In particular $\chi_{\widehat{M}}(x)$ is a non-negative integer.

(c) We have $\chi_{\widehat{M}}(x) \neq 0$ if and only if $x$ belongs to a vertex of $M$.

**Proof:**

(a) Let $S \leqslant G$ be a vertex of $M$. Then $M \mid \mathrm{Ind}_S^G(k)$ and by Mackey's formula any indecomposable direct summand of $\mathrm{Res}_Q^G(M)$ is of the form

$$\mathrm{Ind}_{Q\cap {}^gS}^Q \mathrm{Res}_{Q\cap {}^gS}^G(k) = \mathrm{Ind}_{Q\cap {}^gS}^Q(k)$$

for some $g \in G$. Moreover,

$$\dim_k(\mathrm{soc}(\mathrm{Ind}_{Q\cap {}^gS}^Q(k))) = 1 = \langle \mathrm{Ind}_{Q\cap {}^gS}^Q(1_{Q\cap {}^gS}), 1_Q \rangle_Q$$

where $\mathrm{Ind}_{Q\cap {}^gS}^Q(1_{Q\cap {}^gS})$ is the character afforded by $F \otimes_{\mathcal{O}} \mathrm{Ind}_{Q\cap {}^gS}^Q(\mathcal{O})$. Summing over the indecomposable direct summands of $\mathrm{Res}_Q^G(M)$ yields the claim.

(b) Letting $Q := \langle x \rangle$ in the proof of (a), we obtain

$$\mathrm{Ind}_{Q\cap {}^gS}^Q(1_{Q\cap {}^gS})(x) = \begin{cases} 1 & \text{if } Q \cap {}^gS = Q\,, \\ 0 & \text{otherwise} \end{cases}$$

and the claim follows.

(c) Because $S \in \mathrm{vtx}(M)$, we have $\langle x \rangle \cap {}^gS = \langle x \rangle$ for some $g \in G$ if and only if $x \in {}^gS$, as required. ■

## 34 Brauer Reciprocity

Recall that if $V$ is an $FG$-module, then we can choose an $\mathcal{O}$-form $L$ of $V$ and then consider the reduction modulo $\mathfrak{p}$ of $L$, i.e. $\overline{L} = L/\mathfrak{p}L$. However, the choice of the $\mathcal{O}$-form is not unique. As a consequence, if $L_1$ and $L_2$ are two $\mathcal{O}$-forms of $V$, i.e. $(L_1)^F \cong V \cong (L_2)^F$, then it may happen that $\overline{L_1} \not\cong \overline{L_2}$. The following result shows that, however, this does not affect the composition factors, up to isomorphism and multiplicity.

**Proposition 34.1**

Let $S_1, \ldots, S_t$ ($t \in \mathbb{Z}_{\geqslant 1}$) be a complete set of representatives of the isomorphism classes of simple $kG$-modules. If $V$ is an $FG$-module and $L$ is an $\mathcal{O}$-form of $V$, then for each $1 \leqslant j \leqslant t$ the multiplicity of $S_j$ as a composition factor of $\overline{L} = L/\mathfrak{p}L$ does not depend on the choice of the $\mathcal{O}$-form $L$.

**Proof:** Fix $j \in \{1, \ldots, t\}$. Since $k$ is a splitting field for $G$, $\mathrm{End}_{kG}(S_j) \cong k$. Thus, by Proposition 24.1(b), the multiplicity of $S_j$ as a composition factor of $\overline{L}$ is

$$\dim_k \mathrm{Hom}_{kG}(P_{S_j}, \overline{L}) / \dim_k \mathrm{End}_{kG}(S_j) = \dim_k \mathrm{Hom}_{kG}(P_{S_j}, \overline{L})\,.$$

On the other hand, by Lemma 32.4 and Theorem 32.3, there exists an idempotent $e_j \in \mathcal{O}G$ such that

$$P_{S_j} \cong kG\overline{e}_j\,.$$

Hence, Exercise 32.1(b) yields

$$\mathrm{Hom}_{kG}(P_{S_j}, \overline{L}) \cong \mathrm{Hom}_{kG}(kG\overline{e}_j, \overline{L}) \cong \overline{e}_j\overline{L} \cong \overline{e_j L}\,.$$

Then, Exercise 13.2(a)(ii) and Proposition-Definition 14.6 yield

$$\dim_k \mathrm{Hom}_{kG}(P_{S_j}, \overline{L}) = \dim_k(\overline{e_j L}) = \mathrm{rk}_{\mathcal{O}}(e_j L) = \dim_F(e_j V)\,.$$

As a consequence, for any $1 \leqslant j \leqslant t$, the number of composition factors of $\overline{L}$ isomorphic to $S_j$ is equal to $\dim_F(e_j V)$, and is therefore independent of the choice of the $\mathcal{O}$-form $L$. ■

**Theorem 34.2 (*Brauer Reciprocity*)**

Let $V_1, \ldots, V_l$ ($l \in \mathbb{Z}_{\geqslant 1}$) be a complete set of representatives of the isomorphism classes of simple $FG$-modules, and let $S_1, \ldots, S_t$ ($t \in \mathbb{Z}_{\geqslant 1}$) be a complete set of representatives of the isomorphism classes of simple $kG$-modules. Let $e_1, \ldots, e_t \in \mathcal{O}G$ be idempotents such that $kG\overline{e}_j$ is a projective cover of $S_j$ for each $1 \leqslant j \leqslant t$. For every $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant t$ define $d_{ij}$ to be the multiplicity of $S_j$ as a composition factor of the reduction modulo $\mathfrak{p}$ of an $\mathcal{O}$-form of $V_i$. Then

$$FGe_j \cong \bigoplus_{i=1}^{l} d_{ij} V_i.$$

**Proof:** Since $FG$ is a semisimple $F$-algebra the set $\{V_i \mid 1 \leqslant i \leqslant l\}$ is a complete set of representatives of the isomorphism classes of the PIMs of $FG$ (by Theorem 23.2(b)). Moreover, as $F$ is a splitting field for $G$, by Theorem 8.2, each $V_i$ ($1 \leqslant i \leqslant l$) appears precisely $\dim_F V_i$ times in the decomposition of the regular module $FG$. Hence, for any $1 \leqslant j \leqslant t$, there exist non-negative integers $d'_{ij}$ such that

$$FGe_j = \bigoplus_{i=1}^{l} d'_{ij} V_i\,,$$

where $d'_{ij} = \dim_F \operatorname{Hom}_{FG}(FGe_j, V_i)$. Thus, it remains to prove that $d'_{ij} = d_{ij}$ for every $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant t$. So choose an $\mathcal{O}$-form $L_i$ of $V_i$ ($1 \leqslant i \leqslant l$). As in the previous proof, applying Exercise 32.1 and Proposition 14.6 yields

$$d'_{ij} = \dim_F \operatorname{Hom}_{FG}(FGe_j, V_i) = \dim_F(e_j V_i) = \operatorname{rk}_{\mathcal{O}}(e_j L_i) = \dim_k \overline{e}_j \overline{L}_i = \dim_k \operatorname{Hom}_{kG}(kG\overline{e}_j, \overline{L}_i) = d_{ij}.$$

■

**Definition 34.3 (*Decomposition matrix*)**

The non-negative integers $d_{ij}$ ($1 \leqslant i \leqslant l, 1 \leqslant j \leqslant t$) defined in Theorem 34.2 are called the $p$-**decomposition numbers** of $G$ and the matrix

$$\operatorname{Dec}_p(G) := (d_{ij})_{\substack{1\leqslant i\leqslant l \\ 1\leqslant j\leqslant t}}$$

is called the $p$**-decomposition matrix** (or simply the **decomposition matrix**) of $G$.

**Exercise 34.4**

Let $C$ be the Cartan matrix of $kG$ and put $D := \operatorname{Dec}_p(G)$ for the $p$-decomposition matrix of $G$. Prove that

$$C = D^{tr} D$$

and deduce again that $C$ is symmetric.