Chapter 8. Indecomposable Modules

After simple and projective modules, the goal of this chapter is to understand indecomposable modules in general. Apart for exceptions, the group algebra is of *wild representation type*, which, roughly speaking, means that it is not possible to classify the indecomposable modules over such algebras. However, representation theorists have developed tools which enable us to organise indecomposable modules in packages parametrised by parameters that are useful enough to understand essential properties of these modules. In this respect, first we will generalise the idea of a projective module seen in Chapter 7 by defining what is called **relative projectivity**. This will lead us to introduce the concepts of **vertices** and **sources** of indecomposable modules, which are two typical examples of parameters bringing us useful information about indecomposable modules in general.

Notation: throughout this chapter, unless otherwise specified, we assume Assumption $(*)$ holds.

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27 Relative projectivity

Relative projectivity is a refinement of the idea of projectivity seen in Chapter 7, exploiting induction and restriction from subgroups.

Definition 27.1

Let $H \leqslant G$.

(a) A KG-module M is called **H-free** if there exists a KH-module V such that $M \cong V\uparrow_H^G$.

(b) A KG-module M is called **relatively H-projective**, or simply **H-projective**, if it is isomorphic to a direct summand of an H-free module, i.e. if there exists a KH-module V such that $M \mid V\uparrow_H^G$.

Remark 27.2

It is easy to see that H-freeness is a generalisation of freeness and relative projectivity is a generalisation of projectivity.

(1) Freeness is the same as $\{1\}$-freeness: indeed, as $KG \cong K\uparrow_{\{1\}}^G$ by Example 10, clearly $(KG)^n \cong (K^n)\uparrow_{\{1\}}^G$.

(2) Projectivity is the same as $\{1\}$-projectivity: a KG-module is projective $\Leftrightarrow$ it is a direct summand of a free KG-module $\Leftrightarrow$ it is a direct summand of a $\{1\}$-free KG-module $\Leftrightarrow$ it is relatively $\{1\}$-projective.

To begin with, we would like to characterise relative projectivity in a similar way we characterised projectivity in Proposition-Definition B.5. To reach this aim, we first take a closer look at the adjunction between induction and restriction, we have seen in Theorem 17.10.

Notation 27.3

Let $H \leqslant G$.

(1) Let $\varphi : U_1 \longrightarrow U_2$ be a KH-homomorphism. Then we denote by $\varphi\uparrow_H^G$ the induced KG-homomorhpism

$$\begin{aligned}\varphi\uparrow_H^G := \mathrm{Id}_{KG}\otimes\varphi : U_1\uparrow_H^G = KG\otimes_{KH} U_1 &\longrightarrow U_2\uparrow_H^G = KG\otimes_{KH} U_2\\ x\otimes u &\mapsto x\otimes\varphi(u)\,.\end{aligned}$$

(2) Let U be a KH-module and V be a KG-module. The K-isomorphisms

$$\Phi := \Phi_{U,V} : \mathrm{Hom}_{KG}(U\uparrow_H^G, V) \xrightarrow{\cong} \mathrm{Hom}_{KH}(U, V\downarrow_H^G)$$

and

$$\Psi := \Psi_{U,V} : \mathrm{Hom}_{KH}(U, V\downarrow_H^G) \xrightarrow{\cong} \mathrm{Hom}_{KG}(U\uparrow_H^G, V)$$

from Theorem 17.10 tell us that the induction and restriction functors Ind_H^G and Res_H^G form a pair of bi-adjoint functors. The first isomorphism translates the fact that Ind_H^G is *left adjoint* to Res_H^G and the second isomorphism translates the fact that Ind_H^G is *right adjoint* to Res_H^G.

Explained in more details, there may of course be many such isomorphisms, but there is a choice which is called *natural* in U and V. Spelled out, this means that whenever a morphism $\gamma \in \mathrm{Hom}_{KH}(U_1, U_2)$ is given, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{KG}(U_1\!\uparrow_H^G, V) & \xrightarrow[\cong]{\Phi_{U_1,V}} & \mathrm{Hom}_{KH}(U_1, V\!\downarrow_H^G) \\ {\scriptstyle (\gamma\uparrow_H^G)^*}\uparrow & & \uparrow{\scriptstyle \gamma^*} \\ \mathrm{Hom}_{KG}(U_2\!\uparrow_H^G, V) & \xrightarrow[\Phi_{U_2,V}]{\cong} & \mathrm{Hom}_{KH}(U_2, V\!\downarrow_H^G) \end{array}$$

commutes and whenever $\alpha \in \mathrm{Hom}_{KG}(V_1, V_2)$ is given, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{KH}(U, V_1\!\downarrow_H^G) & \xrightarrow[\cong]{\Psi_{U,V_1}} & \mathrm{Hom}_{KG}(U\!\uparrow_H^G, V_1) \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ \mathrm{Hom}_{KH}(U, V_2\!\downarrow_H^G) & \xrightarrow[\Psi_{U,V_2}]{\cong} & \mathrm{Hom}_{KG}(U\!\uparrow_H^G, V_2) \end{array}$$

commutes. (For the upper and lower $*$ notation, see again Proposition D.3.) For the case Ind_H^G is *right adjoint* to Res_H^G similar diagrams must commute. (Exercise: write down these diagrams!)
In order to understand relative H-projectivity, we consider the **unit** and the **counit** of the adjunction saying that Ind_H^G is *left adjoint* to Res_H^G, i.e. the KH-homomorphism

$$\begin{aligned} \mu : U &\longrightarrow U\!\uparrow_H^G\downarrow_H^G = \bigoplus_{g\in[G/H]} g\otimes U = 1\otimes U \;\oplus \bigoplus_{g\in[G/H], g\neq 1} g\otimes U \\ u &\mapsto 1\otimes u \end{aligned}$$

(i.e. the natural inclusion of U into the summand $1\otimes U$) and the KG-homomorphism

$$\begin{aligned} \varepsilon : V\!\downarrow_H^G\uparrow_H^G = \bigoplus_{g\in[G/H]} g\otimes (V\!\downarrow_H^G) &\longrightarrow V \\ g\otimes v &\mapsto gv\,. \end{aligned}$$

For any $u \in U$, we have $\varepsilon\circ\mu(u) = \varepsilon(1\otimes u) = u$, so $\varepsilon\circ\mu = \mathrm{Id}_U$ and thus we deduce that:

- μ is a KH-section for ε;
- μ is injective; and
- ε is surjective.

This yields the mutually inverse natural K-isomorphisms

$$\Phi = \Phi_{U,V} : \mathrm{Hom}_{KG}(U\!\uparrow_H^G, V) \longrightarrow \mathrm{Hom}_{KH}(U, V\!\downarrow_H^G), \psi \mapsto \psi\circ\mu\,,$$

$$\Psi = \Psi_{U,V} : \mathrm{Hom}_{KH}(U, V\!\downarrow_H^G) \longrightarrow \mathrm{Hom}_{KG}(U\!\uparrow_H^G, V), \beta \mapsto \varepsilon\circ\beta\!\uparrow_H^G\,.$$

Proposition 27.4 (*Characterisation of relative projectivity*)

Let $H \leqslant G$. Let U be a KG-module. Then the following assertions are equivalent.

(a) The KG-module U is relatively H-projective.

(b) If $\psi : U \longrightarrow W$ is a KG-homomorphism, $\varphi : V \twoheadrightarrow W$ is a surjective KG-homomorphism and there exists a KH-homomorphism $\alpha_H : U\downarrow_H^G \longrightarrow V\downarrow_H^G$ such that $\varphi \circ \alpha_H = \psi$ on $U\downarrow_H^G$, then there exists a KG-homomorphism $\alpha_G : U \longrightarrow V$ such that $\varphi \circ \alpha_G = \psi$ so that the diagram on the right commutes.

$$\begin{array}{ccc} & & U \\ & \overset{\exists\,\alpha_G}{\swarrow}\ \circlearrowleft & \big\downarrow \psi \\ V & \underset{\varphi}{\twoheadrightarrow} & W \end{array}$$

(c) Whenever $\varphi : V \twoheadrightarrow U$ is a surjective KG-homomorphism such that the restriction $\varphi : V\downarrow_H^G \longrightarrow U\downarrow_H^G$ splits as KH-homomorphism, then φ splits as a KG-homomorhpism.

(d) The surjective KG-homomorphism

$$\begin{aligned} U\downarrow_H^G\uparrow_H^G = KG \otimes_{KH} U &\longrightarrow U \\ x \otimes u &\mapsto xu \end{aligned}$$

is split.

(e) The KG-module U is a direct summand of $U\downarrow_H^G\uparrow_H^G$.

(f) There exists a KG-module N such that $U \mid K\uparrow_H^G \otimes_K N$.

Proof:

(a)⇒(b): First we consider the case in which $U = T\uparrow_H^G$ is an induced module. Suppose that we have KG-homomorphisms $\psi : T\uparrow_H^G \longrightarrow W$ and $\varphi : V \twoheadrightarrow W$ as shown in the diagram shown on the left below. Suppose, moreover, that there exists a KH-homomorhpism $\alpha_H : T\uparrow_H^G\downarrow_H^G \longrightarrow V\downarrow_H^G$ such that $\psi = \varphi \circ \alpha_H$, that is, the diagram on the right below commutes:

$$\begin{array}{ccc} & & T\uparrow_H^G \\ & & \big\downarrow \psi \\ V & \underset{\varphi}{\twoheadrightarrow} & W \end{array} \qquad\qquad \begin{array}{ccc} & & T\uparrow_H^G\downarrow_H^G \\ & \overset{\exists\,\alpha_H}{\swarrow}\ \circlearrowleft & \big\downarrow \psi \\ V\downarrow_H^G & \underset{\varphi}{\twoheadrightarrow} & W\downarrow_H^G \end{array}$$

Let $\mu : T \longrightarrow T\uparrow_H^G\downarrow_H^G$ and $\varepsilon : V\downarrow_H^G\uparrow_H^G \longrightarrow V$ be the unit and the counit of the adjunction of Res_H^G and Ind_H^G as defined in Notation 27.3, so μ is an injective KH-homomorphism and ε is a surjective KG-homomorphism. Then, precomposing with μ, we obtain that the following triangle of KH-modules and KH-homomorphisms commutes:

$$\begin{array}{ccc} & & T \\ & \overset{\alpha_H\circ\mu}{\swarrow}\ \circlearrowleft & \big\downarrow \psi\circ\mu \\ V\downarrow_H^G & \underset{\varphi}{\twoheadrightarrow} & W\downarrow_H^G \end{array}$$

By the naturality of Ψ from Notation 27.3, since $\varphi : V \longrightarrow W$ is a KG-homomorphism, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{KH}(T, V\downarrow_H^G) & \xrightarrow[\cong]{\Psi_{T,V}} & \mathrm{Hom}_{KG}(T\uparrow_H^G, V) \\ \varphi_*\downarrow & & \downarrow\varphi_* \\ \mathrm{Hom}_{KH}(T, W\downarrow_H^G) & \xrightarrow[\Psi_{T,W}]{\cong} & \mathrm{Hom}_{KG}(T\uparrow_H^G, W) \end{array}$$

In other words,

$$\Psi(\varphi \circ (\alpha_H \circ \mu)) = \varphi \circ (\Psi(\alpha_H \circ \mu)) .$$

By the commutativity of the previous triangle, the left hand side of this equation is equal to $\Psi(\psi \circ \mu) = \Psi(\Phi(\psi)) = \psi$ since Ψ and Φ are inverse to one another. Thus

$$\psi = \varphi \circ \varepsilon \circ ((\alpha_H \circ \mu)\uparrow_H^G)$$

and so the triangle of KG-homomorphisms

$$\begin{array}{ccc} & & T\uparrow_H^G \\ & \swarrow^{\varepsilon\circ((\alpha_H\circ\mu)\uparrow_H^G)} \ \circlearrowleft & \downarrow\psi \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

commutes, proving the implication for $U = T\uparrow_H^G$ an induced module.

Now let U be any direct summand of $T\uparrow_H^G$. Let $U \xrightarrow{\iota} T\uparrow_H^G \xrightarrow{\pi} U$ denote the canonical inclusion and projection. Suppose that there is a KH-homomorphism $\alpha_H : U\downarrow_H^G \longrightarrow V\downarrow_H^G$ such that the diagram

$$\begin{array}{ccc} & & U\downarrow_H^G \\ & \swarrow^{\exists\,\alpha_H} \ \circlearrowleft & \downarrow\psi \\ V\downarrow_H^G & \xrightarrow[\varphi]{} & W\downarrow_H^G \end{array}$$

commutes, i.e. $\varphi \circ \alpha_H = \psi$ on $U\downarrow_H^G$. Then we consider the following diagrams:

$$\begin{array}{ccc} & & T\uparrow_H^G \\ & & \downarrow\psi\circ\pi \\ V & \xrightarrow[\varphi]{} & W \end{array} \qquad \begin{array}{ccc} & & T\uparrow_H^G\downarrow_H^G \\ & \swarrow^{\alpha_H\circ\pi} \ \circlearrowleft & \downarrow\psi\circ\pi \\ V\downarrow_H^G & \xrightarrow[\varphi]{} & W\downarrow_H^G \end{array} \qquad \begin{array}{ccc} & & T\uparrow_H^G \\ & \swarrow^{\alpha_G} \ \circlearrowleft & \downarrow\psi\circ\pi \\ V\downarrow_H^G & \xrightarrow[\varphi]{} & W \end{array}$$

The middle diagram of KH-homomorphisms commutes by definition of α_H, and hence by the first part there is a KG-homomorphism $\alpha_G : T\uparrow_H^G \longrightarrow V$ such that $\varphi \circ \alpha_G = \psi \circ \pi$, so the third diagram of KG-homomorhpisms also commutes.

Now $\varphi \circ \alpha_G \circ \iota = \psi \circ \pi \circ \iota = \psi$, so the triangle

$$\begin{array}{ccc} & & U \\ & \swarrow^{\alpha_G\circ\iota} \ \circlearrowleft & \downarrow\psi \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

commutes, as required.

(b)⇒(c): Let $\varphi : V \twoheadrightarrow U$ be a surjective KG-homomorphism which is split as a KH-homomorphism, and let α_H be a KH-section for φ. Thus, we have the following commutative diagram of KH-modules:

$$\begin{array}{ccc} & & U\downarrow^G_H \\ & \overset{\alpha_H}{\swarrow} \;\circlearrowleft & \Big\downarrow \mathrm{Id}_U \\ V\downarrow^G_H & \underset{\varphi}{\twoheadrightarrow} & U\downarrow^G_H \end{array}$$

Then assuming (b) is true, there exists a KG-homomorphism $\alpha_G : U \longrightarrow V$ such that $\varphi \circ \alpha_G = \mathrm{Id}_U$. In particular, α_G is a KG-section for φ.

(c)⇒(d): Since $\mu : U \longrightarrow U\downarrow^G_H\uparrow^G_H$ is a KH-section for $\varepsilon : U\downarrow^G_H\uparrow^G_H \longrightarrow U$ (see Notation 27.3), applying condition (c) yields that ε splits as a KG-homomorhpism, and hence (d) holds.

(d)⇒(e): Immediate.

(e)⇒(f): Recall that by Proposition 17.11 we have $K\uparrow^G_H \otimes_K N \cong (K \otimes_K N\downarrow^G_H)\uparrow^G_H \cong N\downarrow^G_H\uparrow^G_H$. Thus, setting $N := U$ yields the claim.

(f)⇒(a): This is straightforward from the fact that $K\uparrow^G_H \otimes_K N \cong N\downarrow^G_H\uparrow^G_H$ seen above. ∎

Exercise 27.5

Let $H \leqslant J \leqslant G$. Let U be a KG-module and let V be a KJ-module. Prove the following statements.

(a) If U is H-projective then U is J-projective.

(b) If U is a direct summand of $V\uparrow^G_J$ and V is H-projective, then U is H-projective.

(c) For any $g \in G$, U is H-projective if and only if ${}^gU$ is ${}^gH$-projective.

(d) Using part (f) of Proposition 27.4, prove that if U is H-projective and W is any KG-module, then $U \otimes_K W$ is H-projective.

Projectivity relative to a subgroup can be generalised as follows to projectivity relative to a KG-module:

Remark 27.6 (*Projectivity relative to KG-modules*)

(a) Let V be a KG-module. A KG-module M is termed *projective relative to the module V* or *relatively V-projective*, or simply *V-projective* if there exists a KG-module N such that M is isomorphic to a direct summand of $V \otimes_K N$, i.e. $M \mid V \otimes_K N$.
We let $\mathrm{Proj}(V)$ denote the class of all V-projective KG-modules.

(b) Proposition 27.4(f) shows that projectivity relative to a subgroup $H \leqslant G$ is in fact projectivity relative to the KG-module $V := K\uparrow^G_H$.

Note that the concept of projectivity relative to a subgroup is proper to the group algebra, but the concept of projectivity relative to a module is not and makes sense in general over algebras/rings.

The following exercise provides us with some elementary properties of projectivity relative to a module, which also hold for projectivity relative to a subgroup, by part (b) of the remark.

Exercise 27.7

Assume K is a field of characteristic $p > 0$ (splitting for G) and let A, B, C, U, V be KG-modules. Prove that:

(a) Any direct summand of a V-projective KG-module is V-projective;

(b) If $U \in \mathrm{Proj}(V)$, then $\mathrm{Proj}(U) \subseteq \mathrm{Proj}(V)$;

(c) If $p \nmid \dim_K(V)$ then any KG-module is V-projective;

(d) $\mathrm{Proj}(V) = \mathrm{Proj}(V^*)$;

(e) $\mathrm{Proj}(U \oplus V) = \mathrm{Proj}(U) \oplus \mathrm{Proj}(V)$;

(f) $\mathrm{Proj}(U) \cap \mathrm{Proj}(V) = \mathrm{Proj}(U \otimes_K V)$;

(g) $\mathrm{Proj}(\bigoplus_{j=1}^{n} V) = \mathrm{Proj}(V) = \mathrm{Proj}(\bigotimes_{j=1}^{m} V)\ \ \forall\, m, n \in \mathbb{Z}_{>0}$;

(h) $C \cong A \oplus B$ is V-projective if and only if both A and B are V-projective;

(i) $\mathrm{Proj}(V) = \mathrm{Proj}(V^* \otimes_K V)$.

After this small parenthesis on projectivity relative to modules, we come back to projectivity relative to subgroups. We investigate further what information this concept brings to the understanding of indecomposable KG-modules in general.

Next we see that any indecomposable KG-module can be seen as a relatively projective module with respect to some subgroup of G.

Theorem 27.8

Let $H \leqslant G$.

(a) If $|G : H|$ is invertible in K, then every KG-module is H-projective.

(b) In particular, if K is a field of characteristic $p > 0$ and H contains a Sylow p-subgroup of G, then every KG-module is H-projective.

Proof: (a) Let V be a KG-module. To prove that V is H-projective, we prove that V satisfies Theorem 27.4(c). So let $\varphi : U \twoheadrightarrow V$ be a surjective KG-homomorphism which splits as a KH-homomorphism. We need to prove that φ splits as a KG-homomorphism.
So let $\sigma : V \longrightarrow U$ be a KH-linear section for φ and set

$$\begin{array}{rccl} \tilde{\sigma} : & V & \longrightarrow & U \\ & v & \mapsto & \frac{1}{|G:H|}\sum_{g\in[G/H]} g^{-1}\sigma(gv). \end{array}$$

We may divide by $|G : H|$ since $|G : H| \in K^\times$ and clearly $\tilde{\sigma}$ is well-defined. Now, if $g' \in G$ and $v \in V$, then

$$\tilde{\sigma}(g'v) = \frac{1}{|G:H|}\sum_{g\in[G/H]} g^{-1}\sigma(gg'v) = g'\frac{1}{|G:H|}\sum_{g\in[G/H]} (gg')^{-1}\sigma(gg'v) = g'\tilde{\sigma}(v)$$

and

$$\varphi\widetilde{\sigma}(v) = \frac{1}{|G:H|}\sum_{g\in[G/H]}\varphi\left(g^{-1}\sigma(gv)\right) \overset{\varphi\, KG\text{-lin.}}{=} \frac{1}{|G:H|}\sum_{g\in[G/H]} g^{-1}\varphi\sigma(gv) = \frac{1}{|G:H|}\sum_{g\in[G/H]} v = v$$

where the last-but-one equality holds because $\varphi\sigma = \mathrm{Id}_V$. Thus $\widetilde{\sigma}$ is a KG-linear section for φ.

(b) This follows immediately from (a). Indeed, if $P \in \mathrm{Syl}_p(G)$ and $H \supseteq P$, then $p \nmid |G : H|$, so $|G : H| \in K^{\times}$. ∎

Considering the case $H = \{1\}$ shows that the previous Theorem is in some sense a generalisation of Maschke's Theorem (Theorem 11.1).

Remark 27.9

Assume that K is a field of characteristic $p > 0$ and $H = \{1\}$ is the trivial subgroup. If H contains a Sylow p-subgroup of G then the Sylow p-subgroups of G are trivial, so $p \nmid |G|$. The theorem then says that all KG-modules are $\{1\}$-projective and hence projective.

We know this already, however! If $p \nmid |G|$ then KG is semisimple by Maschke's Theorem (Theorem 11.1), and so all KG-modules are projective by Example 13(d).

Corollary 27.10

Let $H \leqslant G$ and suppose that $|G : H|$ is invertible in K. Then, a KG-module U is projective if and only if $U{\downarrow}_H^G$ is projective.

Again, this holds in particular if K is a field of characteristic $p \geqslant 0$ and H contains a Sylow p-subgroup of G.

Proof: The necessary condition is given by Proposition 23.1(b). To prove the sufficient condition, suppose that $U{\downarrow}_H^G$ is projective. Then, on the one hand,

$$U{\downarrow}_H^G \mid (KH)^n \qquad \text{for some } n \in \mathbb{Z}_{>0}\,.$$

On the other hand, U is H-projective by Theorem 27.8, and it follows from Proposition 27.4(e) that

$$U \mid U{\downarrow}_H^G{\uparrow}_H^G\,.$$

Hence

$$U \mid U{\downarrow}_H^G{\uparrow}_H^G \mid (KH)^n{\uparrow}_H^G \cong (KG)^n\,,$$

so U is projective. ∎

28 Vertices and sources

We now explain some techniques to understand indecomposable modules better. Vertices and sources are two parameters making this possible.

Theorem 28.1

Let U be an indecomposable KG-module.

(a) There is a unique conjugacy class of subgroups Q of G which are minimal subject to the property that U is Q-projective.

(b) Let Q be a minimal subgroup of G such that U is Q-projective. Then, there exists an indecomposable KQ-module T which is unique, up to conjugacy by elements of $N_G(Q)$, such that U is a direct summand of $T\uparrow_Q^G$. Such a KQ-module T is necessarily a direct summand of $U\downarrow_Q^G$.

Proof:

(a) Suppose that U is both H- and L-projective for subgroups H and L of G. Then, by Proposition 27.4(e),
$$U \mid U\downarrow_H^G\uparrow_H^G \qquad \text{and} \qquad U \mid U\downarrow_L^G\uparrow_L^G\ .$$
So, writing $U\downarrow_H^G\uparrow_H^G = U\oplus V$ (where V is a KG-module), we obtain that
$$U\downarrow_H^G\uparrow_H^G\downarrow_L^G\uparrow_L^G = (U\oplus V)\downarrow_L^G\uparrow_L^G$$
and hence
$$U \mid U\downarrow_L^G\uparrow_L^G \mid U\downarrow_H^G\uparrow_H^G\downarrow_L^G\uparrow_L^G\ .$$
By the Mackey formula and transitivity of induction and restriction, it follows that
$$\begin{aligned} U\downarrow_H^G\uparrow_H^G\downarrow_L^G\uparrow_L^G &= \Big(\big(U\downarrow_H^G\big)\uparrow_H^G\downarrow_L^G\Big)\uparrow_L^G \\ &= \Big(\bigoplus_{g\in[L\backslash G/H]}\big({}^g(U\downarrow_H^G)\downarrow_{L\cap {}^gH}^{{}^gH}\big)\uparrow_{L\cap {}^gH}^L\Big)\uparrow_L^G \\ &= \bigoplus_{g\in[L\backslash G/H]}\big({}^gU\downarrow_{L\cap {}^gH}^G\big)\uparrow_{L\cap {}^gH}^G\ . \end{aligned}$$
Therefore, by the Krull-Schmidt Theorem, there exists $g\in G$ such that U is a direct summand of a module induced from $L\cap {}^gH$, and hence U is $L\cap {}^gH$-projective. Now, if both L and H are minimal such that U is projective relative to these subgroups, then $L\cap {}^gH = L$. Thus, $L\subseteq {}^gH$ and $H\subseteq {}^{g^{-1}}L$, hence H and L are G-conjugate.

(b) By the assumption, $U \mid U\downarrow_Q^G\uparrow_Q^G$ by Proposition 27.4(e) so there exists an indecomposable direct summand T of $U\downarrow_Q^G$ such that $U \mid T\uparrow_Q^G$. If T' is another indecomposable KQ-module such that $U \mid T'\uparrow_Q^G$, then $T \mid T'\uparrow_Q^G\downarrow_Q^G$. The Mackey formula says that
$$T'\uparrow_Q^G\downarrow_Q^G = \bigoplus_{g\in[Q\backslash G/Q]}\big({}^gT'\downarrow_{Q\cap {}^gQ}^{{}^gQ}\big)\uparrow_{Q\cap {}^gQ}^Q\,,$$
hence, again by the Krull-Schmidt Theorem, there exists $g\in G$ such that
$$T \mid \big({}^gT'\downarrow_{Q\cap {}^gQ}^{{}^gQ}\big)\uparrow_{Q\cap {}^gQ}^Q$$
and therefore T is $Q\cap {}^gQ$-projective, and hence so is U. Since Q is a minimal subgroup relative to which U is projective, $Q = Q\cap {}^gQ$ and hence $g\in N_G(Q)$. It follows that T is actually a direct summand of ${}^gT'$, for this $g\in G$. Since T and T' are indecomposable, however, this means that $T = {}^gT'$, so T is unique up to conjugacy by elements of $N_G(Q)$.

Now $T = {}^gT'$ is an idecomposable direct summand of $U\downarrow_Q^G$ by definition, so $T' = {}^{g^{-1}}T$ is a direct summand of $({}^{g^{-1}}U)\downarrow_Q^G$. However, $U\cong {}^{g^{-1}}U$ as KG-modules, so this means that T' is also a direct summand of $U\downarrow_Q^G$.

■

This characterisation leads us to the following definition.

Definition 28.2

Let U be an indecomposable KG-module.

(a) A **vertex** of U is a minimal subgroup Q of G such that U is relatively Q-projective.
The set of all vertices of U is denoted by $\text{vtx}(U)$.

(b) Given a vertex Q of U, a KQ**-source**, or simply a **source** of U is a KQ-module T such that $U \mid T\uparrow_Q^G$.

Remark 28.3

(a) Conceptually, the closer the vertices of a module are to the trivial subgroup, the closer this module is to being projective: a KG-module U with trivial vertex is $\{1\}$-projective and hence projective.

(b) A vertex Q of an indecomposable KG-module U is not uniquely defined, in general. However, the vertices of U are unique **up to** G**-conjugacy**, so in particular are all isomorphic. For this reason, in general, one (i.e. you!) should **never** talk about *the* vertex of a module (of course, unless a vertex has been fixed). We either say that Q *is a vertex of* U, or talk about *the vertices of* U. (Unfortunately many textbooks/articles by non-experts are very sloppy with this terminology, inducing errors.)

(c) For a fixed vertex Q of U, a source of U is defined up to conjugacy by elements of $N_G(Q)$.

To begin with, we have the following important restriction on the structure of the vertices.

Proposition 28.4

If K is a field of positive characteristic p, then the vertices of an indecomposable KG-module are p-subgroups of G.

Proof: By Theorem 27.8, we know that every KG-module is projective relative to a Sylow p-subgroup of G. Therefore, by minimality, vertices are contained in Sylow p-subgroups, and hence are themselves p-groups. ■

Warning: vertices and sources are very useful theoretical tools in general, but extremely difficult to compute concretely.

We show here how to compute the vertices and sources of the trivial module.

Example 14

Assume K is a field of positive characteristic p, which is a splitting field for G. Then the vertices of the trivial KG-module K are the Sylow p-subgroups of G, i.e. $\text{vtx}(K) = \text{Syl}_p(G)$, and all sources are trivial.

To establish this fact, we need the following indecomposability property:

Claim: If P is a p-group and $H \leqslant P$, then $K\uparrow_H^P$ is an indecomposable KP-module.

Indeed: First recall that as P is a p-group, the only simple KP-module is the trivial module. Hence the socle of $K\uparrow_H^P$ is a direct sum of trivial submodules. This together with Frobenius reciprocity yields

$$\dim_K \mathrm{soc}(K\uparrow_H^P) = \dim_K \mathrm{Hom}_{KP}(K, K\uparrow_H^P) = \dim_K \mathrm{Hom}_{KH}(K\downarrow_H^P, K) = \dim_K \mathrm{Hom}_{KH}(K, K) = 1\,.$$

If $K\uparrow_H^P$ were decomposable, then so would be its socle: clearly, if $K\uparrow_H^P = U \oplus V$ for some KP-modules $U, V \neq 0$, then

$$\dim_K \mathrm{soc}(K\uparrow_H^P) = \dim_K (\mathrm{soc}(U) \oplus \mathrm{soc}(V)) = \dim_K \mathrm{soc}(U) + \dim_K \mathrm{soc}(V) \geqslant 1 + 1 = 2\,.$$

A contradiction! Therefore $K\uparrow_H^P$ is indecomposable.

Now, let $Q \in \mathrm{vtx}(K)$ and let $P \in \mathrm{Syl}_p(G)$ such that $P \geqslant Q$. Since K is Q-projective,

$$K \mid K\downarrow_Q^G\uparrow_Q^G = K\uparrow_Q^G$$

so

$$K\downarrow_P^G \mid K\uparrow_Q^G\downarrow_P^G = \bigoplus_{g\in[P\backslash G/Q]} K\uparrow_{P\cap {}^gQ}^P$$

by the Mackey formula, and hence, by the Krull-Schmidt Theorem, is a direct summand of $K\uparrow_{P\cap {}^gQ}^P$ for some $g \in G$, which is indecomposable by the Claim. Thus

$$K = K\downarrow_P^G = K\uparrow_{P\cap {}^gQ}^P$$

and hence $P \cap {}^gQ = P$, so ${}^gQ = P$. Therefore, Q is a Sylow p-subgroup of G and it follows from Theorem 28.1(a), that $\mathrm{vtx}(K) = \mathrm{Syl}_p(G)$. Finally, it is clear that the trivial KQ-module is a KQ-source, and hence all sources are trivial.

29 The Green correspondence

The Green correspondence is a correspondence which relates the indecomposable KG-modules with a fixed vertex with the indecomposable KL-modules with the same vertex for well-chosen subgroups $L \leqslant G$. It is used to reduce questions about indecomposable modules to a situation where a vertex of the given indecomposable module is a normal subgroup. This technique is very useful in many situations. In fact, many properties in modular representation theory are believed to be determined by normalisers of p-subgroups.

Lemma 29.1

Let $Q \leqslant G$ be a p-subgroup and let $L \leqslant G$.

(a) If U is an indecomposable KG-module with vertex Q and $L \geqslant Q$, then there exists an indecomposable direct summand of $U\downarrow_L^G$ with vertex Q.

(b) If $L \geqslant N_G(Q)$, then the following assertions hold.

(i) If V is an indecomposable KL-module with vertex Q and U is a direct summand of $V\uparrow_L^G$ such that $V \mid U\downarrow_L^G$, then Q is also a vertex of U.

(ii) If V is an indecomposable KL-module which is Q-projective and there exists an indecomposable direct summand U of $V\uparrow_L^G$ with vertex Q, then Q is also a vertex of V.

Proof: Exercise, Sheet 4. ■

Theorem 29.2 (*Green Correspondence*)

Let Q be a p-subgroup of G and let L be a subgroup of G containing $N_G(Q)$.

(a) If U is an indecomposable KG-module with vertex Q, then

$$U\downarrow_L^G = f(U) \oplus X$$

where $f(U)$ is the unique indecomposable direct summand of $U\downarrow_L^G$ with vertex Q and every indecomposable direct summand of X is $L \cap {}^xQ$-projective for some $x \in G\backslash L$.

(b) If V is an indecomposable KL-module with vertex Q, then

$$V\uparrow_L^G = g(V) \oplus Y$$

where $g(V)$ is unique indecomposable direct summand of $V\uparrow_L^G$ with vertex Q and every indecomposable direct summand of Y is $Q \cap {}^xQ$-projective for some $x \in G\backslash L$.

(c) With the notation of (a) and (b), we then have $g(f(U)) \cong U$ and $f(g(V)) \cong V$. In other words, f and g define a bijection

$$\begin{array}{ccc} \left\{\begin{matrix}\text{isomorphism classes of indecomposable} \\ KG\text{-modules with vertex } Q\end{matrix}\right\} & \xleftrightarrow{\sim} & \left\{\begin{matrix}\text{isomorphism classes of indecomposable} \\ KL\text{-modules with vertex } Q\end{matrix}\right\} \\ U & \mapsto & f(U) \\ g(V) & \leftarrow\!\shortmid & V\ . \end{array}$$

Moreover, corresponding modules have a source in common.

Terminology: $f(U)$ is called the KL-**Green correspondent** of U (or simply the **Green correspondent**) and $g(V)$ is called the KG-**Green correspondent of** V (or simply the **Green correspondent** of V).

Warning! In the Green correspondence it is essential that a vertex Q is fixed and not considered up to conjugation, because the G-conjugacy class of Q and the L-conjugacy class of Q do not coincide in general.

Proof: We first note some properties of the subgroups $Q \cap {}^xQ$ and $L \cap {}^xQ$ for $x \in G\backslash L$:

($*$) Since $N_G(Q) \leqslant L$, x does not normalise Q and hence $Q \cap {}^xQ$ is a proper subgroup of Q.

($**$) $L \cap {}^xQ$ may be of the same order as Q, in which case $L \cap {}^xQ = {}^xQ$.

($***$) Suppose that $L \cap {}^xQ$ is conjugate to Q in L, i.e. there exists $z \in L$ such that $L \cap {}^xQ = {}^zQ$. Then ${}^xQ = {}^zQ$ so ${}^{z^{-1}x}Q = Q$ and hence $z^{-1}x \in N_G(Q) \leqslant L$. Therefore $x \in zL = L$. This contradicts

$x \in G \setminus L$. Therefore $L \cap {}^xQ$ is never conjugate to Q in L.

(b) **Claim.** The KL-module $V\uparrow_L^G\downarrow_L^G$ has a unique direct summand with vertex Q, and all other indecomposable direct summands are projective relative to subgroups of the form $L \cap {}^xQ$ with $x \in G\setminus L$.

Pf of the Claim: Let T be a KQ-source for V. Then, we may write $T\uparrow_Q^L = V \oplus Z$ for some KL-module Z. Moreover, there exist KL-modules V' and Z' such that $V\uparrow_L^G\downarrow_L^G = V \oplus V'$ and $Z\uparrow_L^G\downarrow_L^G = Z \oplus Z'$. Then, on the one hand we have

$$T\uparrow_Q^G\downarrow_L^G = (V \oplus Z)\uparrow_L^G\downarrow_L^G = V\uparrow_L^G\downarrow_L^G \oplus Z\uparrow_L^G\downarrow_L^G = V \oplus V' \oplus Z \oplus Z'.$$

On the other hand, by the Mackey formula we also have

$$\begin{aligned} T\uparrow_Q^G\downarrow_L^G &\cong \bigoplus_{x\in[L\setminus G/Q]} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ} \\ &= T\uparrow_Q^L \oplus \bigoplus_{\substack{x\in[L\setminus G/Q]\\ x\notin L}} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ} \\ &= V \oplus Z \oplus \bigoplus_{\substack{x\in[L\setminus G/Q]\\ x\notin L}} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ} \; . \end{aligned}$$

Therefore, by the Krull-Schmidt Theorem, we have

$$V' \oplus Z' \cong \bigoplus_{\substack{x\in[L\setminus G/Q]\\ x\notin L}} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ}$$

where, clearly, all indecomposable direct summands are $L \cap {}^xQ$-projective for some $x \notin L$. It follows that V is the unique indecomposable direct summand of $V\uparrow_L^G\downarrow_L^G = V \oplus V'$ with vertex Q, because all the direct summands in V' are projective relative to subgroups of the form $L \cap {}^xQ$ with $x \notin L$ and so are not Q-projective by $(***)$. This proves the Claim.

Now, write $V\uparrow_L^G$ as a direct sum of indecomposable KG-modules and pick an indecomposable direct summand U such that $V \mid U\downarrow_L^G$. By Lemma 29.1(b), since Q is a vertex of V, Q is also a vertex of U. Therefore $V\uparrow_L^G$ has at least one indecomposable direct summand with vertex Q. To prove its uniqueness, assume U' is another indecomposable direct summand of $V\uparrow_L^G$. Then

$$V\uparrow_L^G = U \oplus U' \oplus X$$

for some KG-module X, so in the notation of the claim,

$$V \oplus V' = U\downarrow_L^G \oplus U'\downarrow_L^G \oplus X\downarrow_L^G \; .$$

As $V \mid U\downarrow_L^G$, by the Krull-Schmidt-Theorem, $U'\downarrow_L^G \mid V'$ and hence every indecomposable direct summand of $U'\downarrow_L^G$ is $L \cap {}^yQ$-projective for some $y \in G\setminus L$ by the proof of the Claim. Now since $V \mid T\uparrow_Q^L$ and $U' \mid V\uparrow_L^G$ it follows that

$$U' \mid T\uparrow_Q^L\uparrow_L^G = T\uparrow_Q^G \; .$$

Thus U' is Q-projective and therefore has a vertex Q' contained in Q.

It remains to prove that $Q' \lneq Q$. So let S be a KQ'-source of U'. Then $S \mid U'\downarrow_{Q'}^G$ by Theorem 28.1(b). Since $Q' \leqslant L$, $U'\downarrow_{Q'}^G = U'\downarrow_L^G\downarrow_{Q'}^L$ and hence S is a direct summand of $Y\downarrow_{Q'}^L$ for some indecomposable direct summand Y of $U'\downarrow_L^G$. It follows from Lemma 29.1 that Q' is also a vertex of Y. But the indecomposable direct summands of $U'\downarrow_L^G$ are all $L \cap {}^yQ$-projective for some $y \in G\setminus L$. Therefore one of the subgroups $L \cap {}^yQ$ with $y \in G\setminus L$ contains an L-conjugate of Q'. I.e. ${}^zQ' \subseteq L \cap {}^yQ$ for some $z \in L$. Hence $Q' \subseteq {}^{z^{-1}y}Q$ where $z^{-1}y \notin L$ and so there exists $x \in G\setminus L$ such that $Q' \subseteq Q \cap {}^xQ \lneq Q$ by $(*)$. Therefore, we may set $g(V) := U$ and the assertion follows.

(a) Let T be a KQ-source of U. Then U is a direct summand of $T\uparrow_Q^G = T\uparrow_Q^L\uparrow_L^G$, so there is an indecomposable direct summand V of $T\uparrow_Q^L$ such that $U \mid V\uparrow_L^G$. This means that V is Q-projective, and so Q is a vertex of V by Lemma 29.1(b).

Now, by Lemma 29.1(a), there exists an indecomposable direct summand Y of $U\downarrow_L^G$ with vertex Q. But $U\downarrow_L^G \mid V\uparrow_L^G\downarrow_L^G$ and the Claim says that the only direct summand of $V\uparrow_L^G\downarrow_L^G$ with vertex Q is V. Therefore $Y \cong V$ and the remaining indecomposable direct summands of $U\downarrow_L^G$ are $L\cap {}^xQ$-projective for some $x \in G\backslash L$. This proves part (a).

(c) The claim follows immediately from parts (a) and (b) and the facts that $U \mid U\downarrow_L^G\uparrow_L^G$ and $V \mid V\uparrow_L^G\downarrow_L^G$. ■

Exercise 29.3

(a) Verify that modules corresponding to each other via the Green correspondence have a source in common.

(b) Prove that the Green correspondent of the trivial module is the trivial module.

30 p-permutation modules

Assume here that K is a field of positive characteristic p, which is a splitting field for G. Recall from Example 4(b) that any finite G-set X gives rise to a K-representation ρ_X of G. The KG-module corresponding to ρ_X through Proposition 10.3 admits X as K-basis, hence it is standard to denote this module by KX and it is called the **permutation KG-module on X**. Thus, we may pose the following definition.

Definition 30.1 (*Permutation module*)

A KG-module is called a **permutation** KG-module if it admits a K-basis X which is invariant under the action of G. We denote this module by KX.

(Note: it is clear that the basis X is then a finite G-set.)

Permutation KG-modules and, in particular, their indecomposable direct summands have remarkable properties, which we investigate in this section.

Remark 30.2

First, we describe permutation modules and some of their properties more precisely.

If KX is a permutation KG-module on X, then a decomposition of the basis X as a disjoint union of G-orbits, say $X = \bigsqcup_{i=1}^n X_i$, yields a direct sum decomposition of KX as a KG-module as

$$KX = \bigoplus_{i=1}^{n} KX_i\,.$$

Thus, without loss of generality, we can assume that X is a transitive G-set, in which case

$$KX \cong K\uparrow_H^G$$

where $H := \mathrm{Stab}_G(x)$, the stabiliser in G of some $x \in X$. Indeed, clearly we have a direct sum decomposition as a K-vector space

$$KX = \bigoplus_{g \in [G/H]} Kgx$$

and G acts transitively on the summands, so that $KX = K{\uparrow}_H^G$.

It follows that an arbitrary permutation KG-module is isomorphic to a direct sum of KG-modules of the form $K{\uparrow}_H^G$ for various $H \leqslant G$.

Notice that, conversely, an induced module of the form $K{\uparrow}_H^G$ ($H \leqslant G$) is always a permutation KG-module. Indeed, as $K{\uparrow}_H^G = KG \otimes_{KH} K = \bigoplus_{g \in [G/H]} g \otimes K$ as K-vector space, it has on obvious G-invariant K-basis given by the set

$$\{g \otimes 1_K \mid g \in [G/H]\}.$$

In fact, more generally if $H \leqslant G$ and KX is a permutation KH-module on X, then $KX{\uparrow}_H^G$ is a permutation KG-module with G-invariant K-basis $\{g \otimes x \mid g \in [G/H], x \in X\}$. In other words, induction preserves permutation modules.

Exercise 30.3

Prove that direct sums, restriction, inflation and conjugation also preserve permutation modules.

Next, we investigate the indecomposable direct summands of the permutation KG-modules. In order to understand the indecomposable ones, we are going to prove that they all have a trivial source and we will apply the Green correspondence to see that, up to isomorphism, there are only a finite number of them.

Definition 30.4 (*trivial source module*)

A KG-module is called a **trivial source** KG-module if it is indecomposable and has a trivial source K.

Warning: Some texts (books/articles/...) require that a trivial source module is indecomposable, others do not.

Proposition-Definition 30.5 (*p-permutation module*)

Let M be a KG-module and let $P \in \mathrm{Syl}_p(G)$. Then, the following conditions are equivalent:

(a) $M{\downarrow}_Q^G$ is a permutation KQ-module for each p-subgroup $Q \leqslant G$;

(b) $M{\downarrow}_P^G$ is a permutation KP-module;

(c) M has a K-basis which is invariant under the action of P;

(d) M is isomorphic to a direct summand of a permutation KG-module;

(e) M is isomorphic to a direct sum of trivial source KG-modules.

If M fulfils one of these equivalent conditions, then it is called a ***p*-permutation** KG-module.

Note. In fact *p-permutation* KG-modules and *trivial source* KG-modules are two different pieces of terminology for the same concept. French/German speaking authors tend to favour the use of the terminology *p-permutation module* (and reserve the terminology *trivial source module* for an indecomposable module with a trivial source), whereas English speaking authors tend to favour the use of the terminology *trivial source module*.

Proof:

(a)⇔(b): It is obvious that (a) implies (b). For the sufficient condition, notice that for each $g \in G$, we have $M\downarrow^G_{{}^gP} \cong {}^g(M\downarrow^G_P)$. Therefore, as by Exercise 30.3 restriction and conjugation preserve permutation modules, requiring that $M\downarrow^G_P$ is a permutation KP-module implies that $M\downarrow^G_Q$ is a permutation KQ-module for each p-subgroup $Q \leqslant G$. (Because any p-subgroup Q of G is contained in a Sylow p-subgroup and these are all G-conjugate by the Sylow theorems.)

(b)⇔(c): is obvious by the definition of a permutation KP-module.

(b)⇒(e): **Claim**: If L is a KG-module satisfying (b), then so does any direct summand of L.

Proof of the Claim: By Remark 30.2, if $L\downarrow^G_P$ is a permutation KP-module, then there exist $n \in \mathbb{Z}_{n\geqslant 1}$ and subgroups $Q_i \leqslant G$ $(1 \leqslant i \leqslant n)$ such that

$$L\downarrow^G_P \cong \bigoplus_{i=1}^n K\uparrow^P_{Q_i},$$

where each $K\uparrow^P_{Q_i}$ is indecomposable by the Claim in Example 14. Therefore, by the Krull–Schmidt theorem, if $N \mid L$, then $N\downarrow^G_P$ is isomorphic to the direct sum of some of the factors, hence is again a permutation KP-module (by Remark 30.2) and so N satisfies (b) as well, as required.

Now, if M satisfies (b), then by the Claim we can assume w.l.o.g. that M is indecomposable. Let Q be a vertex of M. Then $M \mid M\downarrow^G_Q\uparrow^G_Q$ by Q-projectivity. Since $M\downarrow^G_Q$ is a permutation KQ-module by (a) (⇔ (b)), again by Remark 30.2, there exist $n \in \mathbb{Z}_{\geqslant 0}$ and subgroups $R_i \leqslant Q$ $(1 \leqslant i \leqslant n)$ such that

$$M\downarrow^G_Q \cong \bigoplus_{i=1}^n K\uparrow^Q_{R_i}.$$

Inducing this module to G again and using the Krull-Schmidt theorem, we deduce that M, being indecomposable, is isomorphic to a direct summand of $K\uparrow^G_{R_i}$ for some $1 \leqslant i \leqslant n$. By minimality of the vertex Q, it follows that $R_i = Q$ and that the trivial KQ-module K must be a source of M, proving that M is a trivial source KG-module.

(e)⇒(d): If L is a trivial source module, say with vertex $Q \leqslant G$, then by definition of a source, $L \mid K\uparrow^G_Q$. This implies (d) as $K\uparrow^G_Q$ is a permutation KG-module by Remark 30.2 and any finite direct sum of permutation KG-module is again permutation.

(d)⇒(b): Assume that $M \mid Z$, where Z is a permutation KG-module. Then $M\downarrow^G_P \mid Z\downarrow^G_P$, where $Z\downarrow^G_P$ is again a permutation KP-module by Exercise 30.3. Thus, it follows from the Claim in (b)⇒(e) (see also the scholium below) that $M\downarrow^G_P$ is a permutation KP-module, as required. ■

The Claim in (b)⇒(e) can be formulated as the following result.

Scholium 30.6

If M is a p-permutation KG-module, then any direct summand of M is again a p-permutation KG-module. In particular, if G is a p-group, then any direct summand of a permutation KG-module is a permutation KG-module and so, in this case, any p-permutation module is a permutation module.

Exercise 30.7

Prove that p-permutation modules are preserved by the following operations: direct sums, tensor products, restriction, inflation, conjugation, induction.

Example 15

It is clear that any projective KG-module is a p-permutation KG-module. Also, the PIMs of KG are precisely the KG-modules with vertex $\{1\}$ and trivial source.

Generalising this example, we can characterise the indecomposable p-permutation KG-modules with a given vertex $Q \leqslant G$ as described below.

Example 16

(1) If M is an indecomposable p-permutation KG-module with vertex $Q \leqslant G$, then Q acts trivially on the $KN_G(Q)$-Green correspondent $f(M)$ of M. Thus $f(M)$ can be viewed as a $K[N_G(Q)/Q]$-module. As such, $f(M)$ is indecomposable and projective.

(2) Conversely, if N is a projective indecomposable $K[N_G(Q)/Q]$-module, then $\mathrm{Inf}_{N_G(Q)/Q}^{N_G(Q)}(N)$ is an indecomposable $KN_G(Q)$-module with vertex Q and trivial source. Its KG-Green correspondent is then also an indecomposable KG-module with vertex Q and trivial source, hence is an indecomposable p-permutation KG-module

(3) In this way we obtain a bijection

$$\left\{\begin{matrix}\text{isomorphism classes of indecomposable}\\ p\text{-permutation } KG\text{-modules with vertex } Q\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{isomorphism classes of projective}\\ \text{indecomposable } K[N_G(Q)/Q]\text{-modules}\end{matrix}\right\}.$$

31 Green's indecomposability theorem

To finish our analysis of the indecomposable KG-modules we mention without proof an important indecomposability criterion due to J. A. Green (1959). The proof is rather involved and goes beyond the scope of the techniques we have developed so far.

Theorem 31.1 (*Green's indecomposability criterion, 1959*)

Assume that K is an algebraically closed field of characteristic p. Let $H \leqslant G$ be a subnormal subgroup of G of index a power of p and let M be an indecomposable KH-module. Then $M\uparrow_H^G$ is an indecomposable KG-module.

Proof: Without proof in this lecture. See [Thé95, (23.6) Corollary]. ■

Remark 31.2

Green's indecomposability criterion remains true over an arbitrary field of characteristic p, provided we replace *indecomposability* with *absolute indecomposability*. (A KG-module M is called absolutely indecomposable iff its endomorphism algebra $\mathrm{End}_{KG}(M)$ is a *split local algebra*, that is, if $\mathrm{End}_{KG}(M)/J(\mathrm{End}_{KG}(M)) \cong K$.)

Corollary 31.3

Assume that K is an algebraically closed field of characteristic p. If P is a p-group, $Q \leqslant P$ and M is an indecomposable KQ-module, then $M\uparrow_Q^P$ is an indecomposable KP-module.

Proof: By the Sylow theory, since P is a p-group, any subgroup $Q \leqslant P$ can be plugged in a subnormal series where each quotient is cyclic of order p, hence is a subnormal subgroup of P. Therefore, the claim follows immediately from Green's indecomposability criterion. ■

Notice that, in Example 14, we have proved the latter result in the particular case that $M = K$ is the trivial KQ-module using simple arguments.

Exercise 31.4

Assume that K is an algebraically closed field of characteristic p and let M be an indecomposable KG-module.

(a) Let $Q \in \text{vtx}(M)$ and let $P \in \text{Syl}_p(G)$ such that $Q \leqslant P$. Prove that $|P : Q| \,\big|\, \dim_K(M)$.

(b) Prove that if $\dim_K(M)$ is coprime to p, then $\text{vtx}(M) = \text{Syl}_p(G)$.