Chapter 7. Projective Modules over the Group Algebra

We continue developing techniques to describe modules that are not semisimple and in particular indecomposable modules. The indecomposable projective modules are the indecomposable summands of the regular module. Since every module is a homomorphic image of a direct sum of copies of the regular module, by knowing the structure of the projectives we gain some insight into the structure of all modules.

Notation. Throughout this chapter, unless otherwise specified, we let G denote a finite group. Over a semisimple algebra, any module is projective and for a complete discrete valuation ring $\mathcal{O}$ with residue field k, the projective $\mathcal{O}G$-modules can be recovered from projective kG-modules. For this reason, in this chapter, we simply assume that K is a field. and assume all KG-modules considered are **finitely generated** as KG-modules. When no confusion is to be made, we denote the regular module simply by KG instead of KG°.

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21 Radical, socle, head

Before focusing on projective modules, at this point we examine further the structure of KG-modules which are not semisimple, and try to establish connections with their semisimple submodules, semisimple quotients, and composition factors. This leads us to the definitions of the *radical* and the *socle* of a module.

Definition 21.1

Let M be a KG-module.

(a) The **radical** of M is its submodule $\mathrm{rad}(M) := \bigcap_{V \in \mathrm{Max}(M)} V$ where $\mathrm{Max}(M)$ denotes the set of maximal KG-submodules of M.

(b) The **head** of M is the quotient module $\mathrm{hd}(M) := M/\mathrm{rad}(M)$.

(c) The **socle** of M, denoted $\mathrm{soc}(M)$ is the sum of all simple KG-submodules of M.

Informally (talks/spoken mathematics) one also uses the words *top* and *bottom* instead of *head* and *socle*, respectively.

Lemma 21.2

Let M be a KG-module. Then the following KG-submodules of M are equal:

(1) $\mathrm{rad}(M)$;

(2) $J(KG)M$;

(3) the smallest KG-submodule of M with semisimple quotient.

Proof:

"(3)=(1)": Recall that if $V \in \mathrm{Max}(M)$, then M/V is simple. Moreover, if $V_1, \dots, V_r \in \mathrm{Max}(M)$ ($r \in \mathbb{Z}_{>0}$), then the map

$$\begin{array}{cccc} \varphi: & M & \longrightarrow & M/V_1 \oplus \cdots \oplus M/V_r \\ & m & \mapsto & (m + V_1, \dots, m + V_r) \end{array}$$

is a KG-homomorphism with $\ker(\varphi) = V_1 \cap \cdots \cap V_r$. Hence $M/(V_1 \cap \cdots \cap V_r) \cong \mathrm{Im}(\varphi)$ is semisimple, since it is a KG-submodule of a semisimple KG-module. Therefore $M/\mathrm{rad}(M)$ is a semisimple quotient. It remains to see that it is the smallest such quotient.

If $X \subseteq M$ is a KG-submodule with M/X semisimple, then by the Correspondence Theorem, there exists KG-submodules $X_1, \dots, X_r$ of M ($r \in \mathbb{Z}_{>0}$) containing X such that

$$M/X \cong X_1/X \oplus \cdots \oplus X_r/X \qquad \text{and} \qquad X_i/X \text{ is simple } \forall\, 1 \leqslant i \leqslant r\,.$$

For each $1 \leqslant i \leqslant r$, let Y_i be the kernel of the projection homomorphism $M \twoheadrightarrow M/X \twoheadrightarrow X_i/X$, so that Y_i is maximal (as X_i/X is simple) and $X = Y_1 \cap \ldots \cap Y_r$. Thus $X \supseteq \mathrm{rad}(M)$, as required.

"(1)⊆(2)": Observe that the quotient module $M/J(KG)M$ is a $KG/J(KG)$-module as

$$J(KG)\,(M/J(KG)M) = 0\,.$$

Now, as $KG/J(KG)$ is semisimple (by Proposition 6.6 and Proposition 6.7), $M/J(KG)M$ is a semisimple $KG/J(KG)$-module by definition of a semisimple ring, but then it is also semisimple as a KG-module. Since we have already proved that $\mathrm{rad}(M)$ is the smallest KG-submodule of M with semisimple quotient, we must have that $\mathrm{rad}(M) \subseteq J(KG)M$.

"(2)⊆(1)": If $Z \subseteq M$ is any KG-submodule for which M/Z is semisimple, certainly $J(KG) \cdot M/Z = 0$, because $J(KG)$ annihilates all simple KG-modules by definition, and it follows that $J(KG)M \subseteq Z$. Thus, in particular, we obtain that $J(KG)M \subseteq \mathrm{rad}(M)$. (Again, because we already know that (3)=(1).) ■

Example 12

If M is a semisimple KG-module, then $\mathrm{soc}(M) = M$ by definition, $\mathrm{rad}(M) = 0$ by the above Lemma, and hence $\mathrm{hd}(M) = M$.

Lemma 21.3

Let M be a KG-module. Prove that the following KG-submodules of M are equal:

(1) $\mathrm{soc}(M)$;

(2) the largest semisimple KG-submodule of M;

(3) $\{m \in M \mid J(KG) \cdot m = 0\}$.

Proof: Exercise. [Hint: $\{m \in M \mid J(KG) \cdot m = 0\}$ is the largest KG-submodule of M annihilated by $J(KG)$, and hence may be seen as a $KG/J(KG)$-module.] ■

Remark 21.4 (*Socle, radical and Loewy layers*)

We can iterate the notions of socle and radical: for each KG-module M and each $n \in \mathbb{Z}_{\geqslant 2}$ we define inductively

$$\mathrm{rad}^n(M) := \mathrm{rad}\left(\mathrm{rad}^{n-1}(M)\right) \qquad \text{and} \qquad \mathrm{soc}^n(M)/\mathrm{soc}^{n-1}(M) := \mathrm{soc}(M/\mathrm{soc}^{n-1}(M))$$

where we understand that $\mathrm{rad}^1(M) = \mathrm{rad}(M)$ and $\mathrm{soc}^1(M) = \mathrm{soc}(M)$.

Exercise. Prove that:

(a) $\mathrm{rad}^n(M) = J(KG)^n \cdot M$ and $\mathrm{soc}^n(M) = \{m \in M \mid J(KG)^n \cdot m = 0\}$;

(b) $\cdots \subseteq \mathrm{rad}^3(M) \subseteq \mathrm{rad}^2(M) \subseteq \mathrm{rad}(M) \subseteq M$ and $0 \subseteq \mathrm{soc}(M) \subseteq \mathrm{soc}^2(M) \subseteq \mathrm{soc}^3(M) \subseteq \cdots$

The chains of submodules in (b) are called respectively, the **radical series** and **socle series** of M. The radical series of M is also known as the **Loewy series** of M. The quotients $\mathrm{rad}^{n-1}(M)/\mathrm{rad}^n(M)$ are called the **radical layers**, or **Loewy layers** of M, and the quotients $\mathrm{soc}^n(M)/\mathrm{soc}^{n-1}(M)$ are called the **socle layers** of M.

Exercise 21.5

Let M and N be KG-modules. Prove the following assertions.

(a) For every $n \in \mathbb{Z}_{\geqslant 1}$, $\mathrm{rad}^n(M \oplus N) \cong \mathrm{rad}^n(M) \oplus \mathrm{rad}^n(N)$ and $\mathrm{soc}^n(M \oplus N) \cong \mathrm{soc}^n(M) \oplus \mathrm{soc}^n(N)$.

(b) The radical series of M is the fastest descending series of KG-submodules of M with semisimple quotients, and the socle series of M is the fastest ascending series of M with semisimple quotients. The two series terminate, and if r and n are the least integers for which $\mathrm{rad}^r(M) = 0$ and $\mathrm{soc}^n(M) = M$ then $r = n$.

Definition 21.6

The common length of the radical series and socle series of a KG-module M is called the **Loewy length** of M. (By the above, we may see it as the least integer n such that $J(KG)^n \cdot M = 0$.)

Remark 21.7

The results and arguments used in this section still hold if we assume that K is a commutative ring which is **Artinian**. (Then KG is also left Artinian and left Noetherian, and so are the modules over KG.)

22 Projective modules

For the sake of clarity, we recall the general definition of a projective module through its most standard equivalent characterisations.

Proposition-Definition 22.1 (*Projective module*)

Let R be a ring and let P be an R-module. Then the following are equivalent:

(a) The functor $\mathrm{Hom}_R(P, -)$ is exact. In other words, the image of any s.e.s. of R-modules under $\mathrm{Hom}_R(P, -)$ is again a s.e.s.

(b) If $\psi \in \mathrm{Hom}_R(M, N)$ is a surjective morphism of R-modules, then the morphism of abelian groups $\psi_* : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N)$ is surjective. In other words, for every pair of R-morphisms

$$\begin{array}{ccc} & & P \\ & & \big\downarrow{\scriptstyle\alpha} \\ M & \xrightarrow[\psi]{} & N \end{array}$$

where ψ is surjective, there exists an R-morphism $\beta : P \longrightarrow M$ such that $\alpha = \psi\beta$.

(c) If $\pi : M \longrightarrow P$ is a surjective R-homomorphism, then π splits, i.e., there exists $\sigma \in \mathrm{Hom}_R(P, M)$ such that $\pi \circ \sigma = \mathrm{Id}_P$.

(d) The module P is isomorphic to a direct summand of a free R-module.

If P satisfies these equivalent conditions, then P is called **projective**. Moreover, a projective indecomposable module is called a **PIM** of R.

Example 13

(a) Any free module is projective.

(b) If e is an idempotent element of the ring R, then $R \cong Re \oplus R(1 - e)$ and Re is projective, but not free if $e \neq 0, 1$.

(c) It follows from condition (d) of Proposition-Definition 22.1 that a direct sum of modules $\{P_i\}_{i \in I}$ is projective if and only if each P_i is projective.

(d) If R is semisimple, then on the one hand any projective indecomposable module is simple, and conversely, since R° is semisimple. It follows that any R-module is projective.

23 Projective modules for the group algebra

We have seen that over a semisimple ring, all simple modules appear as direct summands of the regular module with multiplicity equal to their dimension. For non-semisimple rings this is not true any more, but replacing simple modules by the *projective* modules, we will obtain a similar characterisation.

To begin with we review a series of properties of projective KG-modules with respect to the operations on groups and modules we have introduced in Chapter 4, i.e. induction/restriction, tensor products, ...

Proposition 23.1

Here we may assume $K \in \{\mathcal{O}, k\}$.

(a) If P is a projective KG-module and M is an arbitrary KG-module which is free of finite rank as a K-module, then $P \otimes_K M$ is projective.

(b) If P is a projective KG-module and $H \leqslant G$, then $P{\downarrow}_H^G$ is a projective KH-module.

(c) If $H \leqslant G$ and P is a projective KH-module, then $P{\uparrow}_H^G$ is a projective KG-module.

Proof:

(a) Since P is projective, by definition it is a direct summand of a free KG-module, so there exist a KG-module P' and a positive integer n such that $P \oplus P' \cong (KG)^n$. Therefore,

$$(KG)^n \otimes_K M \cong (P \oplus P') \otimes_K M \cong P \otimes_K M \oplus P' \otimes_K M$$

and it suffices to prove that $(KG)^n \otimes_K M$ is free. So observe that Example 10(a), Proposition 17.11(a) and the properties of the tensor product yield

$$\begin{aligned} KG \otimes_K M &\cong (K{\uparrow}_{\{1\}}^G) \otimes_K M \cong (K \otimes_K M{\downarrow}_{\{1\}}^G){\uparrow}_{\{1\}}^G \cong M{\uparrow}_{\{1\}}^G \\ &\cong (K^{\mathrm{rk}_K(M)}){\uparrow}_{\{1\}}^G \cong (K{\uparrow}_{\{1\}}^G)^{\mathrm{rk}_K(M)} \cong (KG)^{\mathrm{rk}_K(M)} \end{aligned}$$

since $M{\downarrow}_{\{1\}}^G$ is just M seen as K-module, and, as such, is free of finite rank. It follows immediately that $(KG)^n \otimes_K M \cong (KG)^{n \cdot \mathrm{rk}_K(M)}$ is a free KG-module, as required.

(b) We have already seen that as a KH-module,

$$KG{\downarrow}_H^G \cong KH \oplus \cdots \oplus KH$$

where KH occurs with multiplicity $|G:H|$, so $KG{\downarrow}_H^G$ is a free KH-module. Hence the restriction from G to H of any free KG-module is a free KH-module. Now, by definition $P \mid F$ for some free KG-module F, so that $P{\downarrow}_H^G \mid F{\downarrow}_H^G$ and the claim follows.

(c) Exercise!
[Hint: prove that $KH{\uparrow}_H^G \cong KG$.] ■

We now want to prove that the PIMs of KG are in bijection with the simple KG-modules, and hence that there are a finite number of them, up to isomorphism. We will then be able to deduce from this bijection that each of them occurs in the decomposition of the regular module with multiplicity equal to the dimension of the corresponding simple module.

Theorem 23.2

(a) If P is a projective indecomposable KG-module, then $P/\operatorname{rad}(P)$ is a simple KG-module.

(b) If M is a KG-module and $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ for a projective indecomposable KG-module P, then there exists a surjective KG-homomorphism $\varphi : P \longrightarrow M$. In particular, if M is also projective indecomposable, then $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$ if and only if $M \cong P$.

(c) There is a bijection

$$\begin{array}{ccc} \left\{\substack{\text{projective indecomposable}\\ KG\text{-modules}}\right\}/\cong & \overset{\sim}{\longleftrightarrow} & \operatorname{Irr}(KG) \\ P & \mapsto & P/\operatorname{rad}(P) \end{array}$$

and hence the number of pairwise non-isomorphic PIMs of KG is finite.

Proof:

(a) By Lemma 21.2, $P/\operatorname{rad}(P)$ is semisimple, hence it suffices to prove that it is indecomposable, or equivalently, by Proposition 5.4 that $\operatorname{End}_{KG}(P/\operatorname{rad}(P))$ is a local ring.
Now, if $\varphi \in \operatorname{End}_{KG}(P)$, then by Lemma 21.2, we have

$$\varphi(\operatorname{rad}(P)) = \varphi(J(KG)P) = J(KG)\varphi(P) \subseteq J(KG)P = \operatorname{rad}(P)\,.$$

Therefore, by the universal property of the quotient, φ induces a unique KG-homomorphism $\overline{\varphi} : P/\operatorname{rad}(P) \longrightarrow P/\operatorname{rad}(P)$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \pi_P\downarrow & \circlearrowleft & \downarrow\pi_P \\ P/\operatorname{rad}(P) & \underset{\overline{\varphi}}{\dashrightarrow} & P/\operatorname{rad}(P) \end{array}$$

Then, the map

$$\begin{array}{cccc} \Phi: & \operatorname{End}_{KG}(P) & \longrightarrow & \operatorname{End}_{KG}(P/\operatorname{rad}(P)) \\ & \varphi & \mapsto & \overline{\varphi} \end{array}$$

is clearly a K-algebra homomorphism. Moreover Φ is surjectiv. Indeed, if $\psi \in \operatorname{End}_{KG}(P/\operatorname{rad}(P))$, then by the definition of a projective module there exists a KG-homomorphism $\varphi : P \longrightarrow P$ such that $\psi \circ \pi_P = \pi_P \circ \varphi$. But then ψ satisfies the diagram of the universal property of the quotient and by uniqueness $\psi = \overline{\varphi}$.
Finally, as P is indecomposable $\operatorname{End}_{KG}(P)$ is local, hence any element of $\operatorname{End}_{KG}(P)$ is either nilpotent or invertible, and by surjectivity of Φ the same holds for $\operatorname{End}_{KG}(P/\operatorname{rad}(P))$, which in turn must be local.

(b) Consider the diagram

$$\begin{array}{ccccc} & & & & P \\ & & & & \downarrow\pi_P \\ M & \underset{\pi_M}{\longrightarrow} & M/\operatorname{rad}(M) & \underset{\psi}{\overset{\cong}{\longrightarrow}} & P/\operatorname{rad}(P) \end{array}$$

where π_M and π_P are the quotient morphisms. As P is projective, by definition, there exists a KG-homomorphism $\varphi : P \longrightarrow M$ such that $\pi_P = \psi \circ \pi_M \circ \varphi$.
It follows that $M = \varphi(P) + \operatorname{rad}(M) = \varphi(P) + J(KG)M$, so that $\varphi(P) = M$ by Nakayama's Lemma.

Finally, if M is a PIM, the surjective homomorphism φ splits by definition of a projective module, and hence $M \mid P$. But as both modules are indecomposable, we have $M \cong P$. Conversely, if $M \cong P$, then clearly $M/\operatorname{rad}(M) \cong P/\operatorname{rad}(P)$.

(c) The given map between the two sets is well-defined by (a) and (b), and it is injective by (b). It remains to prove that it is surjective. So let S be a simple KG-module. As S is finitely generated, there exists a free KG-module F and a surjective KG-homomorphism $\psi : F \longrightarrow S$. But then there is an indecomposable direct summand P of F such that $\psi|_P : P \longrightarrow S$ is non-zero, hence surjective as S is simple. Clearly $\operatorname{rad}(P) \subseteq \ker(\psi|_P)$ since it is the smallest KG-submodule with semisimple quotient by Lemma 21.2. Then the universal property of the quotient yields a surjective homomorphism $P/\operatorname{rad}(P) \longrightarrow S$ induced by $\psi|_P$. Finally, as $P/\operatorname{rad}(P)$ is simple, $P/\operatorname{rad}(P) \cong S$ by Schur's Lemma. ■

Definition 23.3 (*Projective cover of a simple module*)

If S is a simple KG-module, then we denote by P_S the projective indecomposable KG-module corresponding to S through the bijection of Theorem 23.2(c) and call this module the **projective cover** of S.

Corollary 23.4

Assume K is a splitting field for G. In the decomposition of the regular module KG into a direct sum of indecomposable KG-submodules, each isomorphism type of projective indecomposable KG-module occurs with multiplicity

$$\dim_K(P/\operatorname{rad}(P)).$$

In other words, with the notation of Definition 23.3,

$$KG \cong \bigoplus_{S\in \operatorname{Irr}(KG)} (P_S)^{n_S}$$

where $n_S = \dim_K S$.

Proof: Let $KG = P_1 \oplus \cdots \oplus P_r$ ($r \in \mathbb{Z}_{>0}$) be such a decomposition. In particular, $P_1, \dots P_r$ are PIMs. Then

$$J(KG) = J(KG)KG = J(KG)P_1 \oplus \cdots \oplus J(KG)P_r = \operatorname{rad}(P_1) \oplus \cdots \oplus \operatorname{rad}(P_r)$$

by Lemma 21.2. Therefore,

$$KG/J(KG) \cong P_1/\operatorname{rad}(P_1) \oplus \cdots \oplus P_r/\operatorname{rad}(P_r)$$

where each summand is simple by Theorem 23.2(a). Now as $KG/J(KG)$ is semisimple, by Theorem 8.2, any simple $KG/J(KG)$-module occurs in this decomposition with multiplicity equal to its K-dimension. Thus the claim follows from the bijection of Theorem 23.2(c). ■

The Theorem also leads us to the following important dimensional restriction on projective modules.

Corollary 23.5

Assume K is a splitting field for G of characteristic $p > 0$. If P is a projective KG-module, then

$$|G|_p \mid \dim_K(P).$$

(Here $|G|_p$ is the p-part of $|G|$, i.e. the exact power of p that divides the order of G.)

Proof: Let $Q \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. By Lemma 23.1, $P\downarrow_Q^G$ is projective. Moreover, by Corollary 12.4 the trivial KQ-module is the unique simple KQ-module, hence by Theorem 23.2(c) KQ has a unique PIM, namely KQ itself, which has dimension $|Q| = |G|_p$. Hence

$$P\downarrow_Q^G \cong (KQ)^m \qquad \text{for some } m \in \mathbb{Z}_{>0}\,.$$

Therefore,

$$\dim_K(P) = \dim_K(P\downarrow_Q^G) = m \cdot \dim_K KQ = m \cdot |Q| = m \cdot |G|_p$$

and the claim follows. ■

24 The Cartan matrix

Now that we have classified the projective KG-modules we turn to one of their important uses, which is to determine the multiplicity of a simple module S as a composition factor of an arbitrary finitely generated KG-module M (hence with a composition series). We recall that if

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$$

is any composition series of M, the number of quotients M_i/M_{i-1} $(1 \leqslant i \leqslant n)$ isomorphic to S is determined independently of the choice of composition series, by the Jordan–Hölder theorem. We call this number the multiplicity of S in M as a composition factor.

Proposition 24.1

Let $S \in \mathrm{Irr}(KG)$ be a simple KG-module.

(a) If $T \in \mathrm{Irr}(KG)$, then

$$\dim_K \mathrm{Hom}_{KG}(P_S, T) = \begin{cases} \dim_K \mathrm{End}_{KG}(S) & \text{if } S \cong T, \\ 0 & \text{if } S \ncong T. \end{cases}$$

(b) If M is an arbitrary KG-module, then the multiplicity of S as a composition factor of M is

$$\dim_K \mathrm{Hom}_{KG}(P_S, M)/\dim_K \mathrm{End}_{KG}(S)\,.$$

Proof: Exercise, Sheet 4. [Hint: (b) can be proved by induction on the composition length of M.] ■

Definition 24.2

(a) Given $S, T \in \mathrm{Irr}(KG)$, the **Cartan invariant** associated to the pair (S, T) is the non-negative integer

$$c_{ST} := \text{multiplicity of } S \text{ as a composition factor of } P_T\,.$$

(b) The **Cartan matrix** of KG is the matrix $C := (c_{ST})_{S,T\in\mathrm{Irr}(KG)} \in M_{|\mathrm{Irr}(KG)|}(\mathbb{Z})$.

It follows immediately from Proposition 24.1 that the Cartan invariants can be computed as follows.

Corollary 24.3

Let $S, T \in \mathrm{Irr}(KG)$. Then

$$c_{ST} = \dim_K \mathrm{Hom}_{KG}(P_S, P_T)/ \dim_K \mathrm{End}_{KG}(S)\,,$$

and if the base field K is a splitting field for G, then

$$c_{ST} = \dim_K \mathrm{Hom}_{KG}(P_S, P_T)\,.$$

We will see later that there is an extremely effective way of computing the Cartan matrix using another matrix associated to the simple KG-modules, called the *decomposition matrix*.

25 Symmetry of the group algebra

We now want to obtain information about projective KG-modules using duality.
Recall that we have already seen in Lemma 17.9 that $(KG)^* \cong KG$ as (left) KG-modules. This allows us to deduce that projectivity is preserved by taking duals.

Proposition 25.1

Let P be a KG-module. Then, P is projective if and only if P^* is.

Proof: As $P^{**} \cong P$ as KG-modules it suffices to prove one implication. Now, if P is a direct summand of $(KG)^n$ ($n \in \mathbb{Z}_{\geqslant 1}$), then P^* is a direct summand of

$$((KG)^n)^* \cong ((KG)^*)^n \cong (KG)^n\,,$$

and hence is also projective. ■

Next we want to investigate the relationship between head and socle of projective modules. For this purpose, we recall the following properties of submodules, quotients and duality:

$$W \leqslant V\ KG\text{-submodule} \Rightarrow V^* \text{ has a } KG\text{-submodule } M \text{ such that } M \cong (V/W)^* \text{ and } V^*/M \cong W^*\,.$$

Corollary 25.2

Every projective indecomposable KG-module P has a simple socle, more precisely,

$$\mathrm{soc}(P) \cong (P^*/\mathrm{rad}(P^*))^*\,.$$

Proof: As on the one hand $P^*/\mathrm{rad}(P^*)$ is simple by Theorem 23.2 and on the other hand $\mathrm{rad}(P^*)$ is the smallest KG-submodule of P^* with semisimple quotient by Lemma 21.2, its dual is the largest semisimple KG-submodule of $P^{**} \cong P$, hence isomorphic to $\mathrm{soc}(P)$, which has to be simple, as required.

Alternatively, we could argue that as the socle is by definition the sum of all simple submodules, it suffices to prove that P has a unique simple KG-submodule. Because P^* is projective by Proposition 25.1, if S is any simple KG-module, then by duality the KG-homomorphisms $S \longrightarrow P$ are in bijection with the KG-homomorphisms $P^* \longrightarrow S^*$ and it follows that

$$\dim_K \mathrm{Hom}_{KG}(S, P) = \dim_K \mathrm{Hom}_{KG}(P^*, S^*)$$

Moroever S^* is also simple. Thus it follows from Proposition 24.1(a) that

$$\dim_K \mathrm{Hom}_{KG}(P^*, S^*) = \begin{cases} \dim_K \mathrm{End}_{KG}(S^*) & \text{if } P^* \text{ is the projective cover of } S^*, \\ 0 & \text{else.} \end{cases}$$

Therefore, the claim follows from the fact that $\dim_K \mathrm{End}_{KG}(S^*) = \dim_K \mathrm{End}_{KG}(S)$ (again by duality). ■

Notice that in this proof to see that $\dim_K \mathrm{Hom}_{KG}(S, P) = \dim_K \mathrm{Hom}_{KG}(P^*, S^*)$ and $\dim_K \mathrm{End}_{KG}(S^*) = \dim_K \mathrm{End}_{KG}(S)$ it is also possible to argue using Lemma 15.4 and Lemma 16.2.

In fact, we can obtain a more precise statement and prove that the head and the socle of a PIM are isomorphic. For this purpose, we need the fact that the group algebra is a *symmetric algebra*.

Remark 25.3

The map $(\ ,\ ) : G \times G \longrightarrow K$, $(g, h) := \delta_{g,h^{-1}}$ extended by K-bilinearity to

$$(\ ,\ ) : KG \times KG \to K$$

defines a K-bilinear form, which is symmetric, non-degenerate and associative. (Associative means that $(ab, c) = (a, bc)\ \forall\ a, b, c \in KG$).
More generally, a K-algebra endowed with such a symmetric, non-degenerate and associative K-bilinear form is called a **symmetric algebra**.

Theorem 25.4

If P is a projective indecomposable KG-module, then $P/\mathrm{rad}(P) \cong \mathrm{soc}(P)$.

Proof: Put $S := P/\mathrm{rad}(P)$, which we know is simple as P is a PIM of KG, and assume $S \not\cong \mathrm{soc}(P)$. Write $KG = R \oplus Q$, where $Q = P^n$ ($n \in \mathbb{Z}_{>0}$) is the direct sum of all the indecomposable direct summands of KG isomorphic to P and $P \nmid R$. Then

$$\mathrm{soc}(Q) \cong \mathrm{soc}(P)^n$$

and Q does not contain any KG-submodule isomorphic to S. Next, consider the sum of all KG-submodules of KG isomorphic to S and denote it by I, so that clearly $0 \neq I \subset R$ and I is a left ideal of KG. However, as $\psi(I) \subseteq I$ for every $\psi \in \mathrm{End}_{KG}(KG)$, I is an ideal of KG. Now set $J := \{\psi \in \mathrm{End}_{KG}(KG) \mid \mathrm{Im}(\psi) \subset I\}$, so that J is clearly an ideal of $\mathrm{End}_{KG}(KG)$. Let $\pi : KG \to Q$ be the projection onto Q with kernel R. Then:

$$\varphi \in J \Rightarrow \mathrm{rad}(KG) \subseteq \ker(\varphi)$$

as the image of φ is semisimple, because it is a KG-submodule of I. Thus, $\varphi|_R = 0$, as $S \nmid \mathrm{hd}(R)$, and it follows that

$$\varphi \circ \pi = \varphi \text{ and } \pi \circ \varphi = 0$$

(as $\varphi(KG) \subseteq I$) and hence

$$\varphi = \varphi \circ \pi - \pi \circ \varphi \qquad \forall\, \varphi \in J. \qquad (*)$$

Let now $0 \neq \varphi \in J$ (exists since $S = \mathrm{hd}(P) \subseteq \mathrm{hd}(KG)$), and let $\alpha \in \mathrm{End}_{KG}(KG)$, so $\varphi \circ \alpha \in J$ and

$$\varphi\alpha = \varphi\alpha\pi - \pi\varphi\alpha \text{ by } (*).$$

Set then $a := \alpha(1)$, $b := \varphi(1)$, $c := \pi(1)$, so

$$ab = \alpha(1)\varphi(1) = \varphi(\alpha(1)) = \varphi(\alpha(\pi(1))) - \pi(\varphi(\alpha(1))) = cab - abc$$

and it follows from Remark 25.3 that

$$(a, b) = (ab, 1) = (cab, 1) - (abc, 1) = (c, ab) - (ab, c) = 0.$$

This is true for every $\alpha \in \mathrm{End}_{KG}(KG)$, and hence every $a \in KG$. Finally, as $(\ ,\ )$ is non-degenerate, we have $b = 0$ and hence $\varphi = 0$. Contradiction! ■

Corollary 25.5

Let S be a simple KG-module.

(a) If P is any projective KG-module, then the multiplicity of S in $P/\operatorname{rad}(P)$ equals the multiplicity of S in $\operatorname{soc}(P)$. In particular

$$\dim_K(P^G) = \dim_K(P_G) = \dim_K(P^*)^G = \dim_K(P^*)_G\,.$$

(b) We have $(P_S)^* \cong P_{S^*}$.

Proof:

(a) By Theorem 25.4, the first claim holds for the PIMs of KG, hence this is also true for any finite direct sum of PIMs, because taking socles and radicals commute with the direct sum by Exercise 21.5(a). Next taking $S = K$ yields the equalities $\dim_K(P^G) = \dim_K(P_G) = \dim_K(P^*)^G = \dim_K(P^*)_G$.

(b) We have seen in the proof of Corollary 25.2 that $(P_S)^*$ is the projective cover of the simple module $(\operatorname{soc}(P_S))^*$. Moreover, by Theorem 25.4

$$(\operatorname{soc}(P_S))^* \cong (P_S/\operatorname{rad}(P_S))^* \cong S^*\,.$$

Hence $(P_S)^* \cong P_{S^*}$. ■

Finally, we see that the symmetry of the group algebra also leads us to the symmetry of the Cartan matrix.

Theorem 25.6

If S and T are simple KG-modules, then

$$c_{ST} \cdot \dim_K \mathrm{End}_{KG}(S) = c_{TS} \cdot \dim_K \mathrm{End}_{KG}(T)\,.$$

In particular, if K is a splitting field for G, then the Cartan matrix of KG is symmetric.

Proof: By Corollary 24.3,

$$c_{ST} = \dim_K \mathrm{Hom}_{KG}(P_S, P_T)/\dim_K \mathrm{End}_{KG}(S)$$

and

$$c_{TS} = \dim_K \mathrm{Hom}_{KG}(P_T, P_S)/\dim_K \mathrm{End}_{KG}(T)\,,$$

so it is enough to prove that $\dim_K \mathrm{Hom}_{KG}(P_S, P_T) = \dim_K \mathrm{Hom}_{KG}(P_T, P_S)$.
Now, by Lemma 16.2 and Lemma 15.4 we have

$$\mathrm{Hom}_{KG}(P_S, P_T) = \mathrm{Hom}_K(P_S, P_T)^G \cong ((P_S)^* \otimes_K P_T)^G$$

and

$$\mathrm{Hom}_{KG}(P_T, P_S) = \mathrm{Hom}_K(P_T, P_S)^G \cong ((P_T)^* \otimes_K P_S)^G\,.$$

Moreover, as $(P_S)^* \otimes_K P_T$ is projective by Proposition 23.1(a), it follows from Corollary 25.5(a), that

$$\dim_K(((P_S)^* \otimes_K P_T)^G) = \dim_K(((P_S)^* \otimes_K P_T)^*)^G)\,.$$

But $((P_S)^* \otimes_K P_T)^* \cong P_S \otimes_K (P_T)^* \cong (P_T)^* \otimes_K P_S$, thus we have proved that $\dim_K \mathrm{Hom}_{KG}(P_S, P_T) = \dim_K \mathrm{Hom}_{KG}(P_T, P_S)$.
Finally, if K is a splitting field for G, then by definition $\mathrm{End}_{KG}(S) \cong K \cong \mathrm{End}_{KG}(T)$, so that the dimension of both endomorphism algebras is one and we have $c_{ST} = c_{TS}$ and we conclude that the Cartan matrix is symmetric. ■

26 Representations of cyclic groups in positive characteristic

We now describe the representations of a cyclic group $G := Z_n = \langle g \mid g^n = 1\rangle$ of order $n \in \mathbb{Z}_{\geqslant 1}$ over a field K of positive characteristic.

Notation: Set $p := \mathrm{char}(K) > 0$ and write $n = p^a m$ with $a \in \mathbb{Z}_{\geqslant 0}$, $m \in \mathbb{Z}_{\geqslant 1}$ and $\gcd(p, m) = 1$. Moreover, we assume that K is a splitting field for G, so it follows that K contains a primitive m-th root of unity, which we denote by ζ_m. This enables us to use the theory of Jordan normal forms (Linear Algebra).

Theorem 26.1

There are exactly n isomorphism classes of indecomposable KZ_n-modules. These correspond to the n matrix representations

$$R_{i,r} : G \to \mathrm{GL}_r(K), \quad g \mapsto \begin{bmatrix} \zeta_m^i & 1 & & \\ & \zeta_m^i & \ddots & \\ & & \ddots & 1 \\ & & & \zeta_m^i \end{bmatrix} \qquad (1 \leqslant i \leqslant m,\ 1 \leqslant r \leqslant p^a).$$

Proof: First notice that $R_{i,r}$ $(1 \leqslant i \leqslant m, 1 \leqslant r \leqslant p^a)$ defines a matrix representation of $G = Z_n$ since $R_{i,r}(g)^n = I_r$. Furthermore, if $(e_1, \dots, e_r)$ is the standard K-basis of K^r, then the only Z_n-invariant subspaces are the $\langle e_1, \dots, e_j\rangle_K$ with $1 \leqslant j \leqslant r$. As they form a chain, $R_{i,r}$ is indecomposable for all $1 \leqslant i \leqslant m$, $1 \leqslant r \leqslant p^a$, because it cannot be written as the direct sum of two non-trivial subrepresentations. It is also clear that the $R_{i,r}$ are pairwise non-equivalent, as they are uniquely determined through K-dimension and eigenvalues at evaluation in g.
It remains to prove that the $R_{i,r}$ $(1 \leqslant i \leqslant m, 1 \leqslant r \leqslant p^a)$ account for all the indecomposable KZ_n-modules. We know from the theory of Jordan normal form, that if M is a KG-module with $\dim_K(M) =: r \in \mathbb{Z}_{>0}$, then choosing a suitable K-basis, we may assume that M corresponds to a matrix representation R such that $R(g)$ is a block diagonal matrix where each block is a Jordan block. Assuming now that M is indecomposable, then there can be only one Jordan block. Moreover, as

$$R(g)^n = R(g^n) = R(1_G) = I_r\,,$$

the eigenvalues (i.e. the diagonal entries of the Jordan blocks) can only be n-th roots of unity in K, and hence they are powers ζ_m^i $(1 \leqslant i \leqslant m)$ of ζ_m since $\mathrm{char}(K) = p$. Furthermore,

$$R(g) = su = us$$

with $s = \mathrm{diag}(\zeta_m^i, \dots, \zeta_m^i)$ and u is the Jordan block with diagonal entries equal to 1, where it holds that

$$(u - I_r)^{p^s} = u^{p^s} - I_r = 0 \quad \forall\ p^s \geqslant r,$$

so u is the p-part of $R(g)$ and it follows that $1 \leqslant r \leqslant p^a$.

■

Corollary 26.2

Up to isomorphism:

- the simple KZ_n-modules correspond precisely to the matrix representations $R_{i,1}$ $(1 \leqslant i \leqslant m)$;
- the PIMs of KZ_n correspond precisely to the matrix representations R_{i,p^a} $(1 \leqslant i \leqslant m)$.

Proof: We can bound the number of modules in both families of modules as follows.

- Firstly, the matrix representations $R_{i,1}$ $(1 \leqslant i \leqslant m)$ all have degree 1, hence they must be irreducible and correspond to simple KZ_n-modules. It follows that there are at least m pairwise non-isomorphic simple KZ_n-modules.
- Secondly, by Corollary 23.5, if P is a PIM of KZ_n, then $p^a \mid \dim_K(P)$. Hence, up to equivalence, P corresponds to one of the matrix representations R_{i,p^a} with $1 \leqslant i \leqslant m$. It follows that there are at most m pairwise non-isomorphic PIMs.

However, we know from Theorem 23.2(c) that there is a bijection between the simple KZ_n-modules and the PIMs of KZ_n. The claim follows. ■