Chapter 5. Operations on Groups and Modules

In this chapter we show how to construct new modules over the group algebra from old ones using standard module operations such as tensor products, Hom-functors, duality, or using subgroups or quotients of the initial group. Moreover, we study how these constructions relate to each other.

Notation: throughout this chapter, unless otherwise specified, we let G denote a finite group and $(F, \mathcal{O}, k)$ be a splitting p-modular system for G and its subgroups. Moreover, we let $K \in \{F, \mathcal{O}, k\}$ (so that K is in particular always a commutative ring), and we assume that all KG-modules considered are **free of finite rank as K-modules**, hence finitely generated as KG-modules.

References:

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15 Tensors, Hom's and duality

Definition 15.1 (*Tensor product of KG-modules*)

If M and N are two KG-modules, then the tensor product $M \otimes_K N$ of M and N balanced over K becomes a KG-module via the **diagonal action** of G. In other words, the external composition law is defined by the G-action

$$\begin{array}{rccl} \cdot : & G \times (M \otimes_K N) & \longrightarrow & M \otimes_K N \\ & (g, m \otimes n) & \mapsto & g \cdot (m \otimes n) := gm \otimes gn \end{array}$$

extended by K-linearity to the whole of KG.

Definition 15.2 (Homs)

If M and N are two KG-modules, then the abelian group $\mathrm{Hom}_K(M,N)$ becomes a KG-module via the so-called **conjugation action** of G. In other words, the external composition law is defined by the G-action

$$\begin{array}{rccl} \cdot: & G \times \mathrm{Hom}_K(M,N) & \longrightarrow & \mathrm{Hom}_K(M,N) \\ & (g,f) & \mapsto & g\cdot f: M \longrightarrow N, m \mapsto (g\cdot f)(m) := g\cdot f(g^{-1}\cdot m) \end{array}$$

extended by K-linearity to the whole of KG.

Specifying Definition 15.2 to $N = K$ yields a KG-module structure on the K-dual $M^* = \mathrm{Hom}_K(M,K)$.

Definition 15.3 (*Dual of a KG-module*)

(a) If M is a KG-module, then its K-dual M^* becomes a KG-module via the external composition law is defined by the map

$$\begin{array}{rccl} \cdot: & G \times M^* & \longrightarrow & M^* \\ & (g,f) & \mapsto & g\cdot f: M \longrightarrow K, m \mapsto (g\cdot f)(m) := f(g^{-1}\cdot m) \end{array}$$

extended by K-linearity to the whole of KG.

(b) If M, N are KG-modules, then every morphism $\rho \in \mathrm{Hom}_{KG}(M,N)$ induces a KG-homomorphism

$$\begin{array}{rccl} \rho^*: & N^* & \longrightarrow & M^* \\ & f & \mapsto & \rho^*(f): M \longrightarrow K, m \mapsto \rho^*(f)(m) := f\circ\rho(m)\,. \end{array}$$

(See Proposition D.3.)

Lemma 15.4

If M and N are KG-modules, then $\mathrm{Hom}_K(M,N) \cong M^* \otimes_K N$ as KG-modules.

Proof: The map

$$\begin{array}{rccl} \theta := \theta_{M,N}: & M^* \otimes_K N & \longrightarrow & \mathrm{Hom}_K(M,N) \\ & f\otimes n & \mapsto & \theta(f\otimes n): M \longrightarrow N, m \mapsto \theta(f\otimes n)(m) = f(m)n \end{array}$$

defines a K-isomorphism. (Check it!)
Now, for every $g \in G$, $f \in M^*$, $n \in N$ and $m \in M$, we have on the one hand

$$\begin{aligned} \theta(g\cdot(f\otimes n))(m) &= \theta(g\cdot f \otimes g\cdot n))(m) = (g\cdot f)(m) g\cdot n \\ &= f(g^{-1}\cdot m) g\cdot n \end{aligned}$$

and on the other hand

$$\big(g\cdot\theta(f\otimes n)\big)(m) = g\cdot\big(\theta(f\otimes n)(g^{-1}m)\big) = g\cdot\big(f(g^{-1}m)n\big) = f(g^{-1}\cdot m)g\cdot n\,,$$

hence $\theta(g\cdot(f\otimes n)) = \big(g\cdot\theta(f\otimes n)\big)$ and it follows that θ is in fact a KG-isomorphism. ■

Remark 15.5

In case $M = N$ the above constructions yield a KG-module structure on $\mathrm{End}_K(M) \cong M^* \otimes_K M$. Moreover, if $\mathrm{rk}_K(M) =: n$, $\{m_1, \dots, m_n\}$ is a K-basis of M and $\{m_1^*, \dots, m_n^*\}$ is the dual K-basis, then $\mathrm{Id}_M \in \mathrm{End}_K(M)$ corresponds to the element $\mathrm{r} := \sum_{i=1}^n m_i^* \otimes m_i \in M^* \otimes_K M$. (Exercise!)
This allows us to define the KG-homomorphism

$$\mathbf{I}: \begin{array}{ccc} K & \longrightarrow & M^* \otimes_K M \\ 1 & \mapsto & \mathrm{r}\,. \end{array}$$

Definition 15.6 (*Trace map*)

If M is a KG-module, then the **trace map** associated with M is the KG-homomorphism

$$\mathrm{Tr}_M: \begin{array}{ccc} M^* \otimes_K M & \longrightarrow & K \\ f \otimes m & \mapsto & f(m)\,. \end{array}$$

Exercise 15.7

Let M and N be KG-modules. Prove the following assertions:

(a) $M \cong (M^*)^*$ as KG-modules (in a natural way);

(b) $M^* \oplus N^* \cong (M \oplus N)^*$ and $M^* \otimes_K N^* \cong (M \otimes_K N)^*$ as KG-modules (in a natural way);

(c) M is simple, resp. indecomposable, resp. semisimple, if and only if M^* is simple, resp. indecomposable, resp. semisimple;

Notation 15.8

If M and N are KG-modules, we shall write $M \mid N$ to mean that M is isomorphic to a direct summand of N.

Lemma 15.9

Let M be a KG-module. If $\mathrm{rk}_K(M) \in K^\times$, then $K \mid M^* \otimes_K M$.

Proof: By Lemma-Definition D.4(c) it suffices to check that $\frac{1}{\mathrm{rk}_K(M)}\mathbf{I}$ is a KG-section for Tr_M, because then $M^* \otimes_K M \cong \ker(\mathrm{Tr}_M) \oplus K$, hence $K \mid M^* \otimes_K M$. So let $\lambda \in K$. Then, using the notation of Remark 15.5, we obtain

$$\begin{aligned}\Big[\mathrm{Tr}_M \circ \frac{1}{\mathrm{rk}_K(M)}\mathbf{I}\Big](\lambda) = \frac{1}{\mathrm{rk}_K(M)}\mathrm{Tr}_M(\lambda r) &= \frac{\lambda}{\mathrm{rk}_K(M)}\mathrm{Tr}_M\Big(\sum_{i=1}^n m_i^* \otimes m_i\Big) \\ &= \frac{\lambda}{\mathrm{rk}_K(M)}\sum_{i=1}^n m_i^*(m_i) \\ &= \frac{\lambda}{\mathrm{rk}_K(M)}\sum_{i=1}^n 1 = \lambda\,,\end{aligned}$$

and hence $\mathrm{Tr}_M \circ \frac{1}{\mathrm{rk}_K(M)}\mathbf{I} = \mathrm{Id}_K$. ■

Exercise 15.10

Let M be a KG-module. Prove the following assertions:

(a) Tr_M is a KG-homomorphism and $\mathrm{Tr}_M \circ \theta_{M,M}^{-1}$ coincides with the ordinary trace of matrices;

(b) if $K = k$, then $M \mid M \otimes_k M^* \otimes_k M$, and if char(k) $\mid \dim_k(M)$, then $M \oplus M \mid M \otimes_k M^* \otimes_k M$.

16 Fixed and cofixed points

Fixed and cofixed points explain why in the previous section we considered tensor products and Hom's over K and not over KG.

Definition 16.1 (*G-fixed points and G-cofixed points*)

Let M be a KG-module.

(a) The G-**fixed points** of M are by definition $M^G := \{m \in M \mid g \cdot m = m \ \forall\, g \in G\}$.

(b) The G-**cofixed points** of M are by definition $M_G := M/(I(KG) \cdot M)$.

In other words:

- M^G is the largest KG-submodule of M on which G acts trivially, and
- M_G is the largest quotient of M on which G acts trivially.

Lemma 16.2

If M, N are KG-modules, then $\mathrm{Hom}_K(M, N)^G = \mathrm{Hom}_{KG}(M, N)$ and $(M \otimes_K N)_G \cong M \otimes_{KG} N$.

Proof: A K-linear map $f : M \longrightarrow N$ is a morphism of KG-modules if and only if $f(g \cdot m) = g \cdot f(m)$ for all $g \in G$ and all $m \in M$, that is if and only if $g^{-1} \cdot f(g \cdot m) = f(m)$ for all $g \in G$ and all $m \in M$. This is exactly the condition that f is fixed under the action of G. Hence $\mathrm{Hom}_K(M, N)^G = \mathrm{Hom}_{KG}(M, N)$.
Second claim: Exercise! ■

Exercise 16.3

Let K be a field and let $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ be a s.e.s. of KG-modules. Prove that if $M \cong L \oplus N$, then the s.e.s. splits.

[Hint: Consider the exact sequence induced by the functor $\mathrm{Hom}_{KG}(N, -)$ (as in Proposition D.3(a)) and use the fact that the modules considered are all finite-dimensional.]

17 Inflation, restriction and induction

In this section we define new module structures from known ones for subgroups, overgroups and quotients, and investigate how these relate to each other.

Remark 17.1

(a) If $H \leqslant G$ is a subgroup, then the inclusion $H \longrightarrow G, h \mapsto h$ can be extended by K-linearity to an injective algebra homomorphism $\iota : KH \longrightarrow KG, \sum_{h\in H} \lambda_h h \mapsto \sum_{h\in H} \lambda_h h$. Hence KH is a K-subalgebra of KG.

(b) Similarly, if $U \trianglelefteq G$ is a normal subgroup, then the quotient homomorphism $G \longrightarrow G/U$, $g \mapsto gU$ can be extended by K-linearity to an algebra homomorphism $\pi : KG \longrightarrow K[G/U]$.

It is clear that we can always perform changes of the base ring using the above homomorphism in order to obtain new module structures. This yields two natural operations on modules over group algebras called *inflation* and *restriction*.

Definition 17.2 (*Restriction*)

Let $H \leqslant G$ be a subgroup. If M is a KG-module, then M may be regarded as a KH-module through a change of the base ring along $\iota : KH \longrightarrow KG$, which we denote by $\mathrm{Res}_H^G(M)$ or simply $M{\downarrow}_H^G$ and call the **restriction** of M from G to H.

Definition 17.3 (*Inflation*)

Let $U \trianglelefteq G$ be a normal subgroup. If M is a $K[G/U]$-module, then M may be regarded as a KG-module through a change of the base ring along $\pi : KG \longrightarrow K[G/U]$, which we denote by $\mathrm{Inf}_{G/U}^G(M)$ and call the **inflation** of M from G/U to G.

Remark 17.4

(a) If $H \leqslant G$ is a subgroup, M is a KG-module and $\rho : G \longrightarrow GL(M)$ is the associated K-representation, then the K-representation associated to $M{\downarrow}_H^G$ is simply the composite morphism

$$H \xrightarrow{\iota} G \xrightarrow{\rho} GL(M)\,.$$

(b) Similarly, if $U \trianglelefteq G$ is a normal subgroup, M is a $K[G/U]$-module and $\rho : G/U \longrightarrow GL(M)$ is the associated K-representation, then the K-representation associated to $\mathrm{Inf}_{G/U}^G(M)$ is simply

$$G \xrightarrow{\pi} G/U \xrightarrow{\rho} GL(M)\,.$$

Lemma 17.5

(a) If $H \leqslant G$ and M_1, M_2 are two KG-modules, then $(M_1 \oplus M_2){\downarrow}_H^G = M_1{\downarrow}_H^G \oplus M_2{\downarrow}_H^G$. If $U \trianglelefteq G$ and M_1, M_2 are two $K[G/U]$-modules, then $\mathrm{Inf}_{G/U}^G(M_1 \oplus M_2) = \mathrm{Inf}_{G/U}^G(M_1) \oplus \mathrm{Inf}_{G/U}^G(M_2)$.

(b) **(Transitivity of restriction)** If $L \leqslant H \leqslant G$ and M is a KG-module, then $M{\downarrow}_H^G{\downarrow}_L^H = M{\downarrow}_L^G$.

(c) If $H \leqslant G$ and M is a KG-module, then $(M^*){\downarrow}_H^G \cong (M{\downarrow}_H^G)^*$. If $U \trianglelefteq G$ and M is a $K[G/U]$-module, then $\mathrm{Inf}_{G/U}^G(M^*) \cong (\mathrm{Inf}_{G/U}^G M)^*$.

Proof: (a) Straightforward from the fact that the external composition law on a direct sum is defined componentwise.

(b) If $\iota_{L,H} : L \longrightarrow H$ denotes the canonical inclusion of L in H, $\iota_{H,G} : H \longrightarrow G$ the canonical inclusion of H in G and $\iota_{L,G} : L \longrightarrow G$ the canonical inclusion of L in G, then

$$\iota_{H,G} \circ \iota_{L,H} = \iota_{L,G} \,.$$

Thus performing a change of the base ring via $\iota_{L,G}$ is the same as performing two successive changes of the base ring via first $\iota_{H,G}$ and then $\iota_{L,H}$. Hence $M{\downarrow}^G_H{\downarrow}^H_L = M{\downarrow}^G_L$.

(c) Straightforward. ■

A third natural operation comes from extending scalars from a subgroup to the initial group.

Definition 17.6 (*Induction*)

Let $H \leqslant G$ be a subgroup and let M be a KH-module. Regarding KG as a (KG, KH)-bimodule, we define the **induction** of M from H to G to be the left KG-module

$$\mathrm{Ind}_H^G(M) := KG \otimes_{KH} M$$

where the KG acts via its left action on itself. We also write $M{\uparrow}_H^G$ instead of $\mathrm{Ind}_H^G(M)$.

Example 10

(a) If $H = \{1\}$ and $M = K$, then $K{\uparrow}_{\{1\}}^G = KG \otimes_K K \cong KG$.

(b) **Transitivity of induction**: clearly $L \leqslant H \leqslant G$ and M is a KL-module, then $M{\uparrow}_L^G = (M{\uparrow}_L^H){\uparrow}_H^G$ by the associativity of the tensor product.

First, we analyse the structure of an induced module in terms of the left cosets of H.

Remark 17.7

Recall that $G/H = \{gH \mid g \in G\}$ denotes the set of left cosets of H in G. Moreover, we write $[G/H]$ for a set of representatives of these left cosets. In other words, $[G/H] = \{g_1, \dots, g_{|G:H|}\}$ (where we assume that $g_1 = 1$) for elements $g_1, \dots, g_{|G:H|} \in G$ such that $g_iH \neq g_jH$ if $i \neq j$ and G is the disjoint union of the left cosets of H, so that

$$G = \bigsqcup_{g\in[G/H]} gH = g_1H \sqcup \ldots \sqcup g_{|G:H|}H \,.$$

It follows that

$$KG = \bigoplus_{g\in[G/H]} gKH \,,$$

where $gKH = \{g\sum_{h\in H}\lambda_h h \mid \lambda_h \in K \ \forall\, h \in H\}$. Clearly, $gKH \cong KH$ as *right* KH-modules via $gh \mapsto h$ for each $h \in H$. Therefore

$$KG \cong \bigoplus_{g\in[G/H]} KH = (KH)^{|G:H|}$$

and hence is a free *right* KH-module with a KH-basis given by the left coset representatives in $[G/H]$.

In consequence, if M is a given KH-module, then we have

$$KG \otimes_{KH} M = \Big(\bigoplus_{g\in[G/H]} gKH\Big) \otimes_{KH} M = \bigoplus_{g\in[G/H]} (gKH \otimes_{KH} M) = \bigoplus_{g\in[G/H]} (g \otimes M)\,,$$

where we set

$$g \otimes M := \{g \otimes m \mid m \in M\} \subseteq KG \otimes_{KH} M\,.$$

Clearly, each $g \otimes M$ is isomorphic to M as a K-module via the K-isomorphism

$$g \otimes M \longrightarrow M, g \otimes m \mapsto m\,.$$

It follows that

$$\mathrm{rk}_K(\mathrm{Ind}_H^G(M)) = |G : H| \cdot \mathrm{rk}_K(M)\,.$$

Next we see that with its left action on $KG \otimes_{KH} M$, the group G permutes these K-submodules: for if $x \in G$, then $xg_i = g_j h$ for some $h \in H$, and hence

$$x \cdot (g_i \otimes m) = xg_i \otimes m = g_j h \otimes m = g_j \otimes hm\,.$$

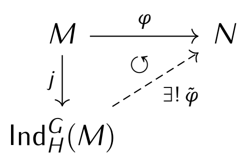
This action is also clearly transitive since for every $1 \leqslant i, j \leqslant |G : H|$ we can write

$$g_j g_i^{-1}(g_i \otimes M) = g_j \otimes M\,.$$

Exercise: Check that the stabiliser of $g_1 \otimes M$ is H (where $g_1 = 1$) and deduce that the stabiliser of $g_i \otimes M$ is $g_i H g_i^{-1}$.

Proposition 17.8 (*Universal property of the induction*)

Let $H \leqslant G$, let M be a KH-module and let $j : M \longrightarrow KG \otimes_{KH} M, m \mapsto 1 \otimes m$ be the canonical map (which is in fact a KH-homomorphism). Then, for every KG-module N and for every KH-homomorphism $\varphi : M \longrightarrow \mathrm{Res}_H^G(N)$, there exists a unique KG-homomorphism $\tilde{\varphi} : KG \otimes_{KH} M \longrightarrow N$ such that $\tilde{\varphi} \circ j = \varphi$, or in other words such that the following diagram commutes:



Proof: The universal property of the tensor product yields the existence of a well-defined homomorphism of abelian groups

$$\begin{array}{rccc} \tilde{\varphi}: & KG \otimes_{KH} M & \longrightarrow & N \\ & a \otimes m & \mapsto & a \cdot \varphi(m)\ . \end{array}$$

which is obviously KG-linear. Moreover, for each $m \in M$, we have $\tilde{\varphi} \circ j(m) = \tilde{\varphi}(1 \otimes m) = 1 \cdot \varphi(m) = \varphi(m)$, hence $\tilde{\varphi} \circ j = \varphi$. Finally the uniqueness follows from the fact for each $a \in KG$ and each $m \in M$, we have

$$\tilde{\varphi}(a \otimes m) = \tilde{\varphi}(a \cdot (1 \otimes m)) = a \cdot \tilde{\varphi}(1 \otimes m) = a \cdot (\tilde{\varphi} \circ j(m)) = a \cdot \varphi(m)$$

hence there is a unique possible definition for $\tilde{\varphi}$. ■

Induced modules can be hard to understand from first principles, so we now develop some formalism that will enable us to compute with them more easily.

To begin with, there is in fact a further operation that relates the modules over a group G and a subgroup H called *coinduction*. Given a KH-module M, then the **coinduction** of M from H to G is the left KG-module

$$\mathrm{Coind}_H^G(M) := \mathrm{Hom}_{KH}(KG, M)$$

where the left KG-module structure is defined through the natural right KG-module structure of KG, i.e. for $g \in G$:

$$\begin{array}{rccl} \cdot: & KG \times \mathrm{Hom}_{KH}(KG, M) & \longrightarrow & \mathrm{Hom}_{KH}(KG, M) \\ & (g, \theta) & \mapsto & (\, g\cdot\theta : KG \longrightarrow M, x \mapsto (g\cdot\theta)(x) := \theta(x\cdot g)\,) \end{array}$$

Example 11

If $H = \{1\}$ and $M = K$, then $\mathrm{Coind}_{\{1\}}^G(K) \cong (KG)^*$ (i.e. with the KG-module structure on $(KG)^*$ of Definition 15.3).

Exercise: exhibit a KG-isomorphism between the coinduction of K from $\{1\}$ to G and $(KG)^*$.

Now, we see that the operation of coinduction in the context of group algebras is just a disguised version of the induction functor.

Lemma 17.9 (*Induction and coinduction are the same*)

If $H \leqslant G$ is a subgroup and M is a KH-module, then $KG \otimes_{KH} M \cong \mathrm{Hom}_{KH}(KG, M)$ as KG-modules. In particular, $KG \cong (KG)^*$ as KG-modules.

Proof: Mutually inverse KG-isomorphisms are defined by:

$$\begin{array}{rccll} \Phi: & KG \otimes_{KH} M & \longrightarrow & \mathrm{Hom}_{KH}(KG, M) & \\ & g \otimes m & \mapsto & \Phi_{g\otimes m} & \text{(for } g \in G,\ m \in M) \end{array}$$

where $\Phi_{g\otimes m} : KG \longrightarrow M$ is such that for $s \in G$, $\Phi_{g\otimes m}(s) := sgm$ if $sg \in H$ and $\Phi_{g\otimes m}(s) := 0$ if $sg \notin H$; and

$$\begin{array}{rccl} \Psi: & \mathrm{Hom}_{KH}(KG, M) & \longrightarrow & KG \otimes_{KH} M \\ & \theta & \mapsto & \sum_{g\in[G/H]} g \otimes \theta(g^{-1})\,. \end{array}$$

It follows that in the case in which $H = \{1\}$ and $M = K$,

$$KG \cong KG \otimes_K K \cong \mathrm{Hom}_K(KG, K) \cong (KG)^*$$

as KG-modules. ∎

Theorem 17.10 (*Adjunction / Frobenius reciprocity / Nakayama relations*)

Let $H \leqslant G$ be a subgroup. Let N be a KG-module and let M be a KH-module. Then, there are K-isomorphisms:

(a) $\mathrm{Hom}_{KH}(M, \mathrm{Hom}_{KG}(KG, N)) \cong \mathrm{Hom}_{KG}(KG \otimes_{KH} M, N)$,
or in other words, $\mathrm{Hom}_{KH}(M, N{\downarrow}_H^G) \cong \mathrm{Hom}_{KG}(M{\uparrow}_H^G, N)$;

(b) $\mathrm{Hom}_{KH}(N{\downarrow}_H^G, M) \cong \mathrm{Hom}_{KG}(N, M{\uparrow}_H^G)$.

Proof: (a) Since induction and coinduction coincide, we have $\mathrm{Hom}_{KG}(KG, N) \cong KG \otimes_{KG} N \cong N$ as KG-modules. Therefore, $\mathrm{Hom}_{KG}(KG, N) \cong N\downarrow^G_H$ as KH-modules, and it suffices to prove the second isomorphism. In fact, this K-isomorphism is given by the map

$$\begin{array}{rccc} \Phi: & \mathrm{Hom}_{KH}(M, N\downarrow^G_H) & \longrightarrow & \mathrm{Hom}_{KG}(M\uparrow^G_H, N) \\ & \varphi & \mapsto & \tilde{\varphi} \end{array}$$

where $\tilde{\varphi}$ is the KG-homomorphism induced by φ by the universal property of the induction. Since $\tilde{\varphi}$ is the unique KG-homomorphism such that $\tilde{\varphi} \circ j = \varphi$, setting

$$\begin{array}{rccc} \Psi: & \mathrm{Hom}_{KG}(M\uparrow^G_H, N) & \longrightarrow & \mathrm{Hom}_{KH}(M, N\downarrow^G_H) \\ & \psi & \mapsto & \psi \circ j \end{array}$$

provides us with an inverse map for Φ. Finally, it is straightforward to check that both Φ and Ψ are K-linear.

(b) Exercise: check that the so-called *exterior trace map*

$$\begin{array}{rccc} \widehat{\mathrm{Tr}}^G_H: & \mathrm{Hom}_{KH}(N\downarrow^G_H, M) & \longrightarrow & \mathrm{Hom}_{KG}(N, M\uparrow^G_H) \\ & \varphi & \mapsto & \widehat{\mathrm{Tr}}^G_H(\varphi): N \longrightarrow M\uparrow^G_H, n \mapsto \sum_{g\in[G/H]} g \otimes \varphi(g^{-1}n) \end{array}$$

provides us with the required K-isomorphism. ■

Proposition 17.11

Let $H \leqslant G$ be a subgroup. Let N be a KG-module and let M be a KH-module. Then, there are KG-isomorphisms:

(a) $(M \otimes_K N\downarrow^G_H)\uparrow^G_H \cong M\uparrow^G_H \otimes_K N$; and

(b) $\mathrm{Hom}_K(M, N\downarrow^G_H)\uparrow^G_H \cong \mathrm{Hom}_K(M\uparrow^G_H, N)$.

Proof: (a) It follows from the associativity of the tensor product that

$$(M \otimes_K N\downarrow^G_H)\uparrow^G_H = KG \otimes_{KH} (M \otimes_K N\downarrow^G_H) \cong (KG \otimes_{KH} M) \otimes_K N = M\uparrow^G_H \otimes_K N$$

(b) We push back the proof until we have introduced the concept of an H-free module. (We will then prove that if M is a KH-module, then $(M^*)\uparrow^G_H \cong (M\uparrow^G_H)^*$ and (b) will follow directly from (a) and the KG-isomorphism of Lemma 15.4.) ■

Exercise 17.12

Let K be a field. Let U, V, W be KG-modules. Prove that there are isomorphisms of KG-modules:

(i) $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(U, V^* \otimes_K W)$; and

(ii) $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W) \cong \mathrm{Hom}_{KG}(U, \mathrm{Hom}_K(V, W))$.