Chapter 3. Representation Theory of Finite Groups

Representation theory of finite groups is originally concerned with the ways of writing a finite group G as a group of matrices, that is, using group homomorphisms from G to the general linear group $\mathrm{GL}_n(K)$ of invertible $n \times n$-matrices with coefficients in a field K for some positive integer n. Thus, we shall first define representations of groups using this approach. Our aim is then to translate such homomorphisms $G \longrightarrow \mathrm{GL}_n(K)$ into the language of module theory in order to be able to apply the theory we have developed so far. In particular, our first aim is to understand what the general theory of semisimple rings and the Artin-Wedderburn theorem bring to the theory of representations of finite groups.

Notation: throughout this chapter, unless otherwise specified, we let G denote a finite group and K be a commutative ring. Moreover, in order to simplify some arguments, we assume that all KG-modules considered are **free of finite rank** when regarded as K-modules. (This implies, in particular, that they are **finitely generated** as KG-modules.)

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9 Linear representations of finite groups

To begin with, we review elementary definitions and examples about representations of finite groups.

Definition 9.1 (*K-representation, matrix representation*)

(a) A **K-representation** of G is a group homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$, where $V \cong K^n$ ($n \in \mathbb{Z}_{\geqslant 0}$) is a free K-module of finite rank and $\mathrm{GL}(V) := \mathrm{Aut}_K(V)$.

(b) A **matrix representation** of G over K is a group homomorphism $X : G \longrightarrow \mathrm{GL}_n(K)$ ($n \in \mathbb{Z}_{\geqslant 0}$).

In both cases the integer n is called the **degree** of the representation.

(c) If K is a field, then a K-representation (resp. a matrix representation) is called an **ordinary representation** if $\mathrm{char}(K) \nmid |G|$, and it is called a **modular representation** if $\mathrm{char}(K) \mid |G|$.

Remark 9.2

Both concepts of a representation and of a matrix representation are closely related. Indeed, choosing a K-basis B of V, then we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}(V) \\ & \overset{\circlearrowleft}{\underset{\exists}{\searrow}} & \updownarrow \cong \\ & & \mathrm{GL}_n(K)\,. \end{array}$$

In other words, any K-representation of a group G defined a matrix representation of G (with respect to the basis B), and conversely.

Example 4

(a) If G is an arbitrary finite group, then

$$\begin{array}{rccl} \rho : & G & \longrightarrow & \mathrm{GL}(K) \cong K^\times \\ & g & \mapsto & \rho(g) := \mathrm{Id}_K \leftrightarrow 1_K \end{array}$$

is a K-representation of G, called <u>**the**</u> **trivial representation** of G.

(b) If X is a finite G-set, i.e. a finite set endowed with a left action $\cdot : G \times X \longrightarrow X$, and V is a free K-module with basis $\{e_x \mid x \in X\}$, then

$$\begin{array}{rccl} \rho_X : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_X(g) : V \longrightarrow V, e_x \mapsto e_{g\cdot x} \end{array}$$

is a K-representation of G, called the **permutation representation** associated with X.

Two particularly interesting examples are the following:

(1) if $G = S_n$ ($n \geqslant 1$) is the symmetric group on n letters and and $X = \{1, 2, \ldots, n\}$ then ρ_X is called **natural representation** of S_n ;

(2) if $X = G$ and the left action $\cdot : G \times X \longrightarrow X$ is just the multiplication in G, then $\rho_X =: \rho_{\mathrm{reg}}$ is called the **regular representation** of G.

Definition 9.3 (*Homomorphism of representations, equivalent representations*)

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two K-representations of G, where V_1, V_2 are two non-zero free K-modules of finite rank.

(a) A K-homomorphism $\alpha : V_1 \longrightarrow V_2$ such that $\rho_2(g) \circ \alpha = \alpha \circ \rho_1(g)$ for each $g \in G$ is called a **homomorphism of representations** (or a G**-homomorphism**) between ρ_1 and ρ_2.

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle\alpha}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle\alpha} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array}$$

(b) If, moreover, α is a K-isomorphism, then it is called an **isomorphism of representations** (or a G**-isomorphism**), and the K-representations ρ_1 and ρ_2 are called **equivalent** (or **similar**, or **isomorphic**). In this case we write $\rho_1 \sim \rho_2$.

Remark 9.4

(a) Equivalent representations have the same degree.

(b) Clearly $\sim$ is an equivalence relation.

(c) In consequence, it essentially suffices to study representations up to equivalence (as it essentially suffices to study groups up to isomorphism).

Definition 9.5 (*G-invariant subspace, irreducibility*)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation of G.

(a) A K-submodule $W \subseteq V$ is called G**-invariant** if

$$\rho(g)\big(W\big) \subseteq W \qquad \forall g \in G\,.$$

(In fact in this case the reverse inclusion holds as well, since for each $w \in W$ we can write $w = \rho(gg^{-1})(w) = \rho(g)\big(\rho(g^{-1})(w)\big) \in \rho(g)\big(W\big)$, hence $\rho(g)\big(W\big) = W$.)

(b) The representation ρ is called **irreducible** if it admits exactly two G-invariant K-submodules, namely 0 and V itself; it is called **reducible** if there exists a proper non-zero G-invariant K-submodule $0 \lneq W \subsetneq V$.

Notice that V itself and the zero K-module 0 are always G-invariant.

Definition 9.6 (*Subrepresentation*)

If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a K-representation and $W \subseteq V$ is a G-invariant K-submodule, then

$$\begin{array}{rccl} \rho_W : & G & \longrightarrow & \mathrm{GL}(W) \\ & g & \mapsto & \rho_W(g) := \rho(g)|_W : W \longrightarrow W \end{array}$$

is called a **subrepresentation** of ρ. (This is clearly again a K-representation of G.)

10 The group algebra and its modules

We now want to be able to see K-representations of a group G as *modules*, and more precisely as *modules* over a K-algebra depending on the group G, which is called the *group algebra*:

Lemma-Definition 10.1 (*Group algebra*)

The **group ring** KG is the ring whose elements are the K-linear combinations $\sum_{g\in G}\lambda_g g$ with $\lambda_g \in K\ \forall g\in G$, and addition and multiplication are given by

$$\sum_{g\in G}\lambda_g g+\sum_{g\in G}\mu_g g=\sum_{g\in G}(\lambda_g+\mu_g)g \quad \text{and} \quad \Big(\sum_{g\in G}\lambda_g g\Big)\cdot\Big(\sum_{h\in G}\mu_h h\Big)=\sum_{g,h\in G}(\lambda_g\mu_h)gh$$

respectively. Thus KG is a K-algebra, which as a K-module is free with basis G. Hence we usually call KG the **group algebra of G over K** rather than simply *group ring*.

Proof: By definition KG is a free K-module with basis G, and the multiplication in G is extended by K-bilinearity to the given multiplication $\cdot : KG\times KG \longrightarrow KG$. It is then straightforward that KG bears both the structures of a ring and of a K-module. Finally, axiom (A3) of K-algebras follows directly from the definition of the multiplication and the commutativity of K. ■

Remark 10.2

Clearly:

- $1_{KG}=1_G$;
- the K-rank of KG is $|G|$;
- KG is commutative if and only if G is an abelian group;
- if K is a field, or more generally (left) Artinian, then KG is a left Artinian ring, so that by Hopkins' Theorem a KG-module is finitely generated if and only if it admits a composition series.

Also notice that since G is a group, the map $KG\longrightarrow KG$ defined by $g\mapsto g^{-1}$ for each $g\in G$ is an anti-automorphism. It follows that any *left* KG-module M may be regarded as a *right* KG-module via the right G-action $m\cdot g := g^{-1}\cdot m$. Thus the sidedness of KG-modules is not usually an issue.

As KG is a K-algebra, we may of course consider modules over KG and we recall that any KG-module is in particular a K-module. Moreover, we adopt the following convention, which is automatically satisfied if K is a field.

Convention: in the sequel all KG-modules considered are assumed to be free of finite rank when regarded as K-modules.

Proposition 10.3

(a) Any K-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ of G gives rise to a KG-module structure on V, where the external composition law is defined by the map

$$\begin{array}{rccc} \cdot : & G \times V & \longrightarrow & V \\ & (g, v) & \mapsto & g \cdot v := \rho(g)(v) \end{array}$$

extended by K-linearity to the whole of KG.

(b) Conversely, every KG-module $(V, +, \cdot)$ defines a K-representation

$$\begin{array}{rccl} \rho_V : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_V(g) : V \longrightarrow V, v \mapsto \rho_V(g) := g \cdot v \end{array}$$

of the group G.

Proof: (a) Since V is a K-module it is equipped with an internal addition $+$ such that $(V, +)$ is an abelian group. It is then straightforward to check that the given external composition law defined above verifies the KG-module axioms.

(b) A KG-module is in particular a K-module for the scalar multiplication defined for all $\lambda \in K$ and all $v \in V$ by

$$\lambda v := (\underbrace{\lambda 1_G}_{\in KG}) \cdot v\,.$$

Moreover, it follows from the KG-module axioms that $\rho_V(g) \in \mathrm{GL}(V)$ and also that

$$\rho_V(g_1 g_2) = \rho_V(g_1) \circ \rho_V(g_2)$$

for all $g_1, g_2 \in G$, hence ρ_V is a group homomorphism. ■

Example 5

Via Proposition 10.3 the trivial representation (Example 4(a)) corresponds to the so-called **trivial KG-module**, that is the commutative ring K itself seen as a KG-module via the G-action

$$\begin{array}{rl} \cdot : & G \times K \longrightarrow K \\ & (g, \lambda) \longmapsto g \cdot \lambda := \lambda \end{array}$$

extended by K-linearity to the whole of KG.

Exercise 10.4

Prove that the regular representation ρ_{reg} of G defined in Exampale 4(b)(2) corresponds to the regular KG-module KG° via Proposition 10.3.

Convention: In the sequel, when no confusion is to be made, we drop the $\circ$-notation to denote the regular KG-module and simply write KG instead of KG°.

Lemma 10.5

Two representations $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ are equivalent if and only if $V_1 \cong V_2$ as KG-modules.

Proof: If $\rho_1 \sim \rho_2$ and $\alpha : V_1 \longrightarrow V_2$ is a K-isomorphism such that $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$, then by Proposition 10.3 for every $v \in V_1$ and every $g \in G$ we have

$$g \cdot \alpha(v) = \rho_2(g)(\alpha(v)) = \alpha(\rho_1(g)(v)) = \alpha(g \cdot v)\,,$$

hence α is a KG-isomorphism. Conversely, if $\alpha : V_1 \longrightarrow V_2$ is a KG-isomorphism, then certainly it is a K-homomorphism and for each $g \in G$ and by Proposition 10.3 for each $v \in V_2$ we have

$$\alpha \circ \rho_1(g) \circ \alpha^{-1}(v) = \alpha(\rho_1(g)(\alpha^{-1}(v))) = \alpha(g \cdot \alpha^{-1}(v)) = g \cdot \alpha(\alpha^{-1}(v)) = g \cdot v = \rho_2(g)(v)\,,$$

hence $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$. ■

Remark 10.6 (*Dictionary*)

More generally, through Proposition 10.3, we may transport terminology and properties from KG-modules to K-representations and conversely.

This lets us build the following translation **dictionary**:

K-REPRESENTATIONS		KG-MODULES
K-representation of G	$\longleftrightarrow$	KG-module
degree	$\longleftrightarrow$	K-rank
homomorphism of K-representations	$\longleftrightarrow$	homomorphism of KG-modules
equivalent K-representations	$\longleftrightarrow$	isomorphism of KG-modules
subrepresentation	$\longleftrightarrow$	KG-submodule
direct sum of representations $\rho_{V_1} \oplus \rho_{V_2}$	$\longleftrightarrow$	direct sum of KG-modules $V_1 \oplus V_2$
irreducible representation	$\longleftrightarrow$	simple (= irreducible) KG-module
the trivial representation	$\longleftrightarrow$	the trivial KG-module K
the regular representation of G	$\longleftrightarrow$	the regular KG-module KG
completely reducible K-representation	$\longleftrightarrow$	semisimple KG-module (= completely reducible)
every K-representation of G is completely reducible	$\longleftrightarrow$	KG is semisimple
...		...

Finally we introduce an ideal of KG which encodes a lot of information about KG-modules.

Proposition-Definition 10.7 (*The augmentation ideal*)

The map $\varepsilon : KG \longrightarrow K, \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \lambda_g$ is an algebra homomorphism, called **augmentation homomorphism (or map)**. Its kernel $\ker(\varepsilon) =: I(KG)$ is an ideal and it is called the **augmentation ideal** of KG. The following statements hold:

(a) $I(KG) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\} = \mathrm{ann}_{KG}(K)$ and if K is a field $I(KG) \supseteq J(KG)$;

(b) $KG/I(KG) \cong K$ as K-algebras;

(c) $I(KG)$ is a free K-module of rank $|G|$-1 with K-basis $\{g-1 \mid g \in G\backslash\{1\}\}$;

Proof: Clearly, the map $\varepsilon : KG \longrightarrow K$ is the unique extension by K-linearity of the trivial representation $G \longrightarrow K^\times \subseteq K, g \mapsto 1_K$ to KG, hence is an algebra homomorphism and its kernel is an ideal of the algebra KG.

(a) $I(KG) = \ker(\varepsilon) = \{\sum_{g\in G} \lambda_g g \in KG \mid \sum_{g\in G} \lambda_g = 0\}$ by definition of ε. The second equality is obvious by definition of $\mathrm{ann}_{KG}(K)$, and the last inclusion follows from the definition of the Jacobson radical.

(b) follows from the 1st isomorphism theorem.

(c) Let $\sum_{g\in G} \lambda_g g \in I(KG)$. Then $\sum_{g\in G} \lambda_g = 0$ and hence

$$\sum_{g\in G} \lambda_g g = \sum_{g\in G} \lambda_g g - 0 = \sum_{g\in G} \lambda_g g - \sum_{g\in G} \lambda_g = \sum_{g\in G} \lambda_g (g-1) = \sum_{g\in G\backslash\{1\}} \lambda_g (g-1),$$

which proves that the set $\{g-1 \mid g \in G\backslash\{1\}\}$ generates $I(KG)$ as a K-module. The above computations also show that

$$\sum_{g\in G\backslash\{1\}} \lambda_g (g-1) = 0 \quad \Longrightarrow \quad \sum_{g\in G} \lambda_g g = 0$$

Hence $\lambda_g = 0 \ \forall\, g \in G$, which proves that the set $\{g-1 \mid g \in G\backslash\{1\}\}$ is also K-linearly independent, hence a K-basis of $I(KG)$. ■

Lemma 10.8

If K is a field of positive characteristic p and G is p-group, then $I(KG) = J(KG)$.

Exercise 10.9 (*Proof of Lemma* 10.8*. Proceed as indicated.*)

(a) Recall that an ideal I of a ring R is called a **nil ideal** if each element of I is nilpotent. Accept the following result: if I is a nil left ideal in a left Artinian ring R then I is nilpotent.

(b) Prove that $g-1$ is a nilpotent element for each $g \in G\backslash\{1\}$ and deduce that $I(KG)$ is a nil ideal of KG.

(c) Deduce from (a) and (b) that $I(KG) \subseteq J(KG)$ using Exercise 2 on Sheet 2.

(d) Conclude that $I(KG) = J(KG)$ using Proposition-Definition 10.7.

11 Semisimplicity and Maschke's Theorem

Throughout this section, we assume that K is a field.

Our first aim is to prove that the semisimplicity of the group algebra depends on both the characteristic of the field and the order of the group.

Theorem 11.1 (*Maschke*)

If $\mathrm{char}(K) \nmid |G|$, then KG is a semisimple K-algebra.

Proof: By Theorem-Definition 6.2, we need to prove that every s.e.s. $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ of KG-modules splits. However, the field K is clearly semisimple (again by Proposition-Definition 6.2). Hence any such sequence regarded as a s.e.s. of K-vector spaces and K-linear maps splits. So let $\sigma : N \longrightarrow M$ be a K-linear section for ψ and set

$$\tilde{\sigma} := \tfrac{1}{|G|}\sum_{g\in G} g^{-1}\sigma g : \quad \begin{array}{ccc} N & \longrightarrow & M \\ n & \mapsto & \tfrac{1}{|G|}\sum_{g\in G} g^{-1}\sigma(gn). \end{array}$$

We may divide by $|G|$, since $\mathrm{char}(K) \nmid |G|$ implies that $|G| \in K^\times$. Now, if $h \in G$ and $n \in N$, then

$$\tilde{\sigma}(hn) = \frac{1}{|G|}\sum_{g\in G} g^{-1}\sigma(ghn) = h\frac{1}{|G|}\sum_{g\in G}(gh)^{-1}\sigma(ghn) = h\tilde{\sigma}(n)$$

and

$$\psi\tilde{\sigma}(n) = \frac{1}{|G|}\sum_{g\in G}\psi\left(g^{-1}\sigma(gn)\right) \overset{\psi\ KG\text{-lin}}{=} \frac{1}{|G|}\sum_{g\in G} g^{-1}\psi\sigma(gn) = \frac{1}{|G|}\sum_{g\in G} g^{-1}gn = n\,,$$

where the last-but-one equality holds because $\psi\sigma = \mathrm{Id}_N$. Thus $\tilde{\sigma}$ is a KG-linear section for ψ. ■

Example 6

If $K = \mathbb{C}$ is the field of complex numbers, then $\mathbb{C}G$ is a semisimple $\mathbb{C}$-algebra, since $\mathrm{char}(\mathbb{C}) = 0$.

It turns out that the converse to Maschke's theorem also holds, and follows from the properties of the augmentation ideal.

Theorem 11.2 (*Converse of Maschke's Theorem*)

If KG is a semisimple K-algebra, then $\mathrm{char}(K) \nmid |G|$.

Proof: Set $\mathrm{char}(K) =: p$ and let us assume that $p \mid |G|$. In particular p must be a prime number. We have to prove that then KG is not semisimple.

Claim: If $0 \neq V \subset KG$ is a KG-submodule of KG°, then $V \cap I(KG) \neq 0$.

Indeed: Let $v = \sum_{g\in G}\lambda_g g \in V\setminus\{0\}$. If $\varepsilon(v) = 0$ we are done. Else, set $t := \sum_{h\in G} h$. Then

$$\varepsilon(t) = \sum_{h\in G} 1 = |G| = 0$$

as $\mathrm{char}(K) \mid |G|$. Hence $t \in I(KG)$. Now consider the element tv. On the one hand $tv \in V$ since V is a submodule of KG°, and on the other hand $tv \in I(KG)\setminus\{0\}$ since

$$tv = \Big(\sum_{h\in G} h\Big)\Big(\sum_{g\in G}\lambda_g g\Big) = \sum_{h,g\in G}(1_K\cdot\lambda_g)hg = \sum_{x\in G}\Big(\sum_{g\in G}\lambda_g\Big)x = \sum_{x\in G}\varepsilon(v)x \Rightarrow \varepsilon(tv) = \sum_{x\in G}\varepsilon(v) = |G|\varepsilon(v) = 0\,.$$

The Claim implies that $I(KG)$, which is a KG-submodule by definition, cannot have a complement in KG°. Therefore, by Proposition 6.1, KG° is not semisimple and hence KG is not semisimple by Theorem-Definition 6.2. ■

In the case in which the field K is algebraically closed, or a splitting field for KG, the following exercise offers a second proof of the converse of Maschke's Theorem exploiting the Artin-Wedderburn Theorem (Theorem 8.2).

Exercise 11.3 (*Proof of the Converse of Maschke's Theorem for K splitting field for KG.*)

Assume K is a field of positive characteristic p with $p \mid |G|$ and is a splitting field for KG. Set $T := \langle \sum_{g\in G} g \rangle_K$.

(a) Prove that we have a series of KG-submodules given by $KG^\circ \supsetneq I(KG) \supseteq T \supsetneq 0$.

(b) Deduce that KG° has at least two composition factors isomorphic to the trivial module K.

(c) Deduce that KG is not a semisimple K-algebra using Theorem 8.2.

12 Simple modules over splitting fields

Assumption 12.1

Throughout this section, we assume that K is a splitting field for KG, and we simply say that K **is a splitting field for** G.

As explained at the end of Chapter 2 this assumption, slightly weaker than requiring that $K = \overline{K}$, implies that the conclusions of Theorem 8.2, Corollary 8.3 and Corollary 8.4 still hold.

We state here some elementary facts about simple KG-modules, which we obtain as consequences of the Artin-Wedderburn structure theorem.

Corollary 12.2

If K is a splitting field for G, then there are only finitely many isomorphism classes of simple KG-modules.

Proof: The claim follows directly from Assumption 12.1 and Corollary 8.3. ■

Corollary 12.3

If G is an abelian group and K is a splitting field for G, then any simple KG-module is one-dimensional.

Proof: Since KG is commutative the claim follows directly from Assumption 12.1 and Corollary 8.4. ■

Corollary 12.4

Let p be a prime number. If G is a p-group, K is a splitting field for G and $\mathrm{char}(K) = p$, then the trivial module is the unique simple KG-module, up to isomorphism.

Proof: By Lemma 10.8 we have $J(KG) = I(KG)$. Thus $KG/J(KG) \cong K$ as K-algebras by Proposition-Definition 10.7(b). Now, as K is commutative, $Z(K) = K$, and it follows from Assumption 12.1 and Corollary 8.3 that

$$|\operatorname{Irr}(KG)| = \dim_K Z(KG/J(KG)) = \dim_K K = 1\,.$$

■

Remark 12.5

Another standard proof for Corollary 12.4 consists in using a result of Brauer's stating that $|\operatorname{Irr}(KG)|$ equals the number of conjugacy classes of G of order not divisible by the characteristic of the field K.

Corollary 12.6

If K is a splitting field for G and $\mathrm{char}(K) \nmid |G|$, then $|G| = \sum_{S\in\operatorname{Irr}(KG)} \dim_K(S)^2$.

Proof: Since $\mathrm{char}(K) \nmid |G|$, the group algebra KG is semisimple by Maschke's Theorem. Thus it follows from Assumption 12.1 and Theorem 8.2 that

$$\sum_{S\in\operatorname{Irr}(KG)} \dim_K(S)^2 = \dim_K(KG) = |G|\,.$$

■