Chapter 2. The Structure of Semisimple Algebras

In this chapter we study an important class of rings: the class of rings R which are such that any R-module can be expressed as a direct sum of *simple* R-submodules. We study the structure of such rings through a series of results essentially due to Artin and Wedderburn. At the end of the chapter we will assume that the ring is a finite dimensional algebra over a field and start the study of its representation theory.

Notation: throughout this chapter, unless otherwise specified, we let R denote a unital and associative ring, and we recall that $\mathrm{Irr}(R)$ denotes a set of representatives for the simple R-modules.

References:

[CR90] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. I.* John Wiley & Sons, Inc., New York, 1990.

[Dor72] L. Dornhoff. *Group representation theory. Part B: Modular representation theory.* Marcel Dekker, Inc., New York, 1972.

[NT89] H. Nagao and Y. Tsushima. *Representations of finite groups.* Academic Press, Inc., Boston, MA, 1989.

[Rot10] J. J. Rotman. *Advanced modern algebra. 2nd ed.* Providence, RI: American Mathematical Society (AMS), 2010.

6 Semisimplicity of rings and modules

To begin with, we prove three equivalent characterisations for the notion of semisimplicity.

Proposition 6.1

If M is an R-module, then the following assertions are equivalent:

(a) M is semisimple, i.e. $M = \bigoplus_{i\in I} S_i$ for some family $\{S_i\}_{i\in I}$ of simple R-submodules of M;

(b) $M = \sum_{i\in I} S_i$ for some family $\{S_i\}_{i\in I}$ of simple R-submodules of M;

(c) every R-submodule $M_1 \subseteq M$ admits a complement in M, i.e. $\exists$ an R-submodule $M_2 \subseteq M$ such that $M = M_1 \oplus M_2$.

Proof:

(a)⇒(b): is trivial.

(b)⇒(c): Write $M = \sum_{i\in I} S_i$, where S_i is a simple R-submodule of M for each $i \in I$. Let $M_1 \subseteq M$ be an R-submodule of M. Then consider the family, partially ordered by inclusion, of all subsets $J \subseteq I$ such that

(1) $\sum_{i\in J} S_i$ is a direct sum, and
(2) $M_1 \cap \sum_{i\in J} S_i = 0$.

Clearly this family is non-empty since it contains the empty set. Thus Zorn's Lemma yields the existence of a maximal element J_0. Now, set

$$M' := M_1 + \sum_{i\in J_0} S_i = M_1 \oplus \sum_{i\in J_0} S_i\,,$$

where the second equality holds by (1) and (2). Therefore, it suffices to prove that $M = M'$, i.e. that $S_i \subseteq M'$ for every $i \in I$. But if $j \in I$ is such that $S_j \nsubseteq M'$, the simplicity of S_j implies that $S_j \cap M' = 0$ and it follows that

$$M' + S_j = M_1 \oplus \left(\sum_{i\in J_0} S_i\right) \oplus S_j$$

in contradiction with the maximality of J_0. The claim follows.

(b)⇒(a): follows from the argument above with $M_1 = 0$.

(c)⇒(b): Let M_1 be the sum of all simple R-submodules in M. By (c) there exists a complement $M_2 \subseteq M$ to M_1, i.e. such that $M = M_1 \oplus M_2$. If $M_2 = 0$, we are done. If $M_2 \neq 0$, then M_2 must contain a simple R-submodule (Exercise: prove this fact), say N. But then $N \subseteq M_1$ by definition of M_1, a contradiction. Thus $M_2 = 0$ and so $M = M_1$. ■

Example 2

(a) The zero module is completely reducible.

(b) If $S_1, \dots, S_n$ are simple R-modules, then their direct sum $S_1 \oplus \dots \oplus S_n$ is completely reducible by definition.

(c) The following exercise shows that there exists modules which are not completely reducible.

<u>Exercise</u>: Let K be a field and let A be the K-algebra $\left\{\left(\begin{smallmatrix} a_1 & a \\ 0 & a_1 \end{smallmatrix}\right) \mid a_1, a \in K\right\}$. Consider the A-module $V := K^2$, where A acts by left matrix multiplication. Prove that:

(1) $\left\{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \mid x \in K\right\}$ is a simple A-submodule of V; but
(2) V is not semisimple.

(d) <u>Exercise</u>: Prove that any submodule and any quotient of a completely reducible module is again completely reducible.

Theorem-Definition 6.2 (*Semisimple ring*)

A ring R satisfying the following equivalent conditions is called **semisimple**.

(a) All short exact sequences of R-modules split.

(b) All R-modules are semisimple.

(c) All finitely generated R-modules are semisimple.

(d) The regular left R-module R° is semisimple, and is a direct sum of a finite number of minimal left ideals.

Proof: First, (a) and (b) are equivalent as a consequence of Lemma ?? and the characterisation of semisimple modules given by Proposition 6.1(c). The implication (b) $\Rightarrow$ (c) is trivial, and it is also trivial that (c) implies the first claim of (d), which in turn implies the second claim of (d). Indeed, if $R^\circ = \bigoplus_{i \in I} L_i$ for some family $\{L_i\}_{i \in I}$ of minimal left ideals. Then, by definition of a direct sum, there exists a finite number of indices $i_1, \ldots, i_n \in I$ such that $1_R = x_{i_1} + \ldots + x_{i_n}$ with $x_{i_j} \in L_{i_j}$ for each $1 \leqslant j \leqslant n$. Therefore each $a \in R$ may be expressed in the form

$$a = a \cdot 1_R = ax_{i_1} + \ldots + ax_{i_n}$$

and hence $R^\circ = L_{i_1} + \ldots + L_{i_n}$.
Therefore, it remains to prove that (d) $\Rightarrow$ (b). So, assume that R satisfies (d) and let M be an arbitrary non-zero R-module. Then write $M = \sum_{m \in M} R \cdot m$. Now, each cyclic submodule $R \cdot m$ of M is isomorphic to an R-submodule of R°, which is semisimple by (d). Thus $R \cdot m$ is semisimple as well by Example 2(d). Finally, it follows from Proposition 6.1(b) that M is semisimple. ■

Example 3

Fields are semisimple. Indeed, if V is a finite-dimensional vector space over a field K of dimension n, then choosing a K-basis $\{e_1, \cdots, e_n\}$ of V yields $V = Ke_1 \oplus \ldots \oplus Ke_n$, where $\dim_K(Ke_i) = 1$, hence Ke_i is a simple K-module for each $1 \leqslant i \leqslant n$. Hence, the claim follows from Theorem-Definition 6.2(c).

Corollary 6.3

Let R be a semisimple ring. Then:

(a) R° has a composition series;

(b) R is both left Artinian and left Noetherian.

Proof:

(a) By Theorem-Definition 6.2(d) the regular module R° admits a direct sum decomposition into a finite number of minimal left ideals. Removing one ideal at a time, we obtain a composition series for R°.

(b) Since R° has a composition series, it satisfies both D.C.C. and A.C.C. on submodules by Corollary 3.4. In other words, R is both left Artinian and left Noetherian. ■

Next, we show that semisimplicity is detected by the Jacobson radical. This leads us to introduce a slightly weaker concept: the notion of *J-semisimplicity*.

Definition 6.4 (*J-semimplicity*)

A ring R is said to be **J-semisimple** if $J(R) = 0$.

Exercise 6.5

Let $R = \mathbb{Z}$. Prove that $J(\mathbb{Z}) = 0$, but not all $\mathbb{Z}$-modules are semisimple. In other words, $\mathbb{Z}$ is J-semisimple but not semisimple.

Proposition 6.6

Any left Artinian ring R is J-semisimple if and only if it is semisimple.

Proof: "$\Rightarrow$": Assume $R \neq 0$ and R is not semisimple. Pick a minimal left ideal $I_0 \trianglelefteq R$ (e.g. a minimal element of the family of non-zero principal left ideals of R). Then $0 \neq I_0 \neq R$ since I_0 seen as an R-module is simple.

Claim: I_0 is a direct summand of R°.

Indeed: since

$$I_0 \neq 0 = J(R) = \bigcap_{\substack{I \triangleleft R,\\ I \text{ maximal}\\ \text{left ideal}}} I$$

there exists a maximal left ideal $\mathfrak{m}_0 \triangleleft R$ which does not contain I_0. Thus $I_0 \cap \mathfrak{m}_0 = \{0\}$ and so we must have $R^\circ = I_0 \oplus \mathfrak{m}_0$, as $R/\mathfrak{m}_0$ is simple. Hence the Claim.

Notice that then $\mathfrak{m}_0 \neq 0$, and pick a minimal left ideal I_1 in $\mathfrak{m}_0$. Then $0 \neq I_1 \neq \mathfrak{m}_0$, else R would be semisimple. The Claim applied to I_1 yields that I_1 is a direct summand of R°, hence also in $\mathfrak{m}_0$. Therefore, there exists a non-zero left ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 = I_1 \oplus \mathfrak{m}_1$. Iterating this process, we obtain an infinite descending chain of ideals

$$\mathfrak{m}_0 \supsetneq \mathfrak{m}_1 \supsetneq \mathfrak{m}_2 \supsetneq \cdots$$

contradicting D.C.C. and proving the claim.

"$\Leftarrow$": Conversely, if R is semisimple, then $R^\circ \cong R/J(R) \oplus J(R)$ by Theorem-Definition 6.2 and so as R-modules,

$$J(R) = J(R) \cdot (R/J(R) \oplus J(R)) = J(R) \cdot J(R)$$

so that by Nakayama's Lemma $J(R) = 0$.

■

Proposition 6.7

The quotient ring $R/J(R)$ is J-semisimple.

Proof: Since by Exercise 4.2 the rings R and $\overline{R} := R/J(R)$ have the same simple modules (seen as abelian groups), Proposition-Definition 4.1(a) yields

$$J(\overline{R}) = \bigcap_{V \in \mathrm{Irr}(\overline{R})} \mathrm{ann}_{\overline{R}}(V) = \bigcap_{V \in \mathrm{Irr}(R)} \mathrm{ann}_R(V) + J(R) = J(R)/J(R) = 0\,.$$

■

7 The Artin-Wedderburn structure theorem

The next step in analysing semisimple rings and modules is to sort simple modules into isomorphism classes. We aim at proving that each isomorphism type of simple modules actually occurs as a direct summand of the regular module. The first key result in this direction is the following proposition:

Proposition 7.1

Let M be a semisimple R-module. Let $\{M_i\}_{i\in I}$ be a set of representatives of the isomorphism classes of simple R-submodules of M and for each $i \in I$ set

$$H_i := \sum_{\substack{V \subseteq M \\ V \cong M_i}} V \,.$$

Then the following statements hold:

(i) $M \cong \bigoplus_{i\in I} H_i$;

(ii) every simple R-submodule of H_i is isomorphic to M_i ;

(iii) $\mathrm{Hom}_R(H_i, H_{i'}) = \{0\}$ if $i \neq i'$; and

(iv) if $M = \bigoplus_{j\in J} V_j$ is an arbitrary decomposition of M into a direct sum of simple submodules, then

$$\widetilde{H}_i := \sum_{\substack{j\in J \\ V_j \cong M_i}} V_j = \bigoplus_{\substack{j\in J \\ V_j \cong M_i}} V_j = H_i \,.$$

Proof: We shall prove several statements which, taken together, will establish the theorem.

Claim 1: If $M = \bigoplus_{j\in J} V_j$ as in (iv) and W is an arbitrary simple R-submodule of M, then $\exists\, j \in J$ such that $W \cong V_j$.
Indeed: if $\{\pi_j : M = \bigoplus_{j\in J} V_j \longrightarrow V_j\}_{j\in J}$ denote the canonical projections on the j-th summand, then $\exists\, j \in J$ such that $\pi_j(W) \neq 0$. Hence $\pi_j|_W : W \longrightarrow V_j$ is an R-isomorphism as both W and V_j are simple.

Claim 2: If $M = \bigoplus_{j\in J} V_j$ as in (iv), then $M = \bigoplus_{i\in I} \widetilde{H}_i$ and for each $i \in I$, every simple R-submodule of $\widetilde{H}_i$ is isomorphic to M_i.
Indeed: the 1st statement of the claim is obvious and the 2nd statement follows from Claim 1 applied to $\widetilde{H}_i$.

Claim 3: If W is an arbitrary simple R-submodule of M, then there is a unique $i \in I$ such that $W \subseteq \widetilde{H}_i$.
Indeed: it is clear that there is a unique $i \in I$ such that $W \cong M_i$. Now consider $w \in W \setminus \{0\}$ and write $w = \sum_{j\in J} w_j \in \bigoplus_{j\in J} V_j$ with $w_j \in V_j$. The proof of Claim 1 shows that if any summand $w_j \neq 0$, then $\pi_j(W) \neq 0$, and hence $W \cong V_j$. Therefore $w_j = 0$ unless $V_j \cong M_i$, and hence $w \in \widetilde{H}_i$, so that $W \subseteq \widetilde{H}_i$.

Claim 4: $\mathrm{Hom}_R(\widetilde{H}_i, \widetilde{H}_{i'}) = \{0\}$ if $i \neq i'$.
Indeed: if $0 \neq f \in \mathrm{Hom}_R(\widetilde{H}_i, \widetilde{H}_{i'})$ and $i \neq i'$, then there must exist a simple R-submodule W of $\widetilde{H}_i$ such that $f(W) \neq 0$, hence as W is simple, $f|_W : W \longrightarrow f(W)$ is an R-isomorphism. It follows from Claim 2, that $f(W)$ is a simple R-submodule of $\widetilde{H}_{i'}$ isomorphic to M_i. This contradicts Claim 2 saying that every simple R-submodule of $\widetilde{H}_{i'}$ is isomorphic to $M_{i'} \ncong M_i$.

Now, it is clear that $\widetilde{H}_i \subseteq H_i$ by definition. On the other hand it follows from Claim 3, that $H_i \subseteq \widetilde{H}_i$. Hence $H_i = \widetilde{H}_i$ for each $i \in I$, hence (iv). Then Claim 2 yields (i) and (ii), and Claim 4 yields (iii). ■

We give a name to the submodules $\{H_i\}_{i\in I}$ defined in Propostion 7.1:

Definition 7.2

If M is a semisimple R-module and S is a simple R-module, then the **S-homogeneous component** of M, denoted $S(M)$, is the sum of all simple R-submodules of M isomorphic to S.

Exercise 7.3

Let R be a semisimple ring. Prove the following statements.

(a) Every non-zero left ideal I of R is generated by an **idempotent** of R, in other words $\exists\ e \in R$ such that $e^2 = e$ and $I = Re$. (Hint: choose a complement I' for I, so that $R^\circ = I \oplus I'$ and write $1 = e + e'$ with $e \in I$ and $e' \in I'$. Prove that $I = Re$.)

(b) If I is a non-zero left ideal of R, then every morphism in $\mathrm{Hom}_R(I, R^\circ)$ is given by right multiplication with an element of R.

(c) If $e \in R$ is an idempotent, then $\mathrm{End}_R(Re) \cong (eRe)^{\mathrm{op}}$ (the opposite ring) as rings via the map $f \mapsto ef(e)e$. In particular $\mathrm{End}_R(R^\circ) \cong R^{\mathrm{op}}$ via $f \mapsto f(1)$.

(d) A left ideal Re generated by an idempotent e of R is minimal (i.e. simple as an R-module) if and only if eRe is a division ring. (Hint: Use Schur's Lemma.)

(e) Every simple left R-module is isomorphic to a minimal left ideal in R, i.e. a simple R-submodule of R°.

We recall that:

Definition 7.4 (*Centre*)

The **centre** of a ring $(R, +, \cdot)$ is $Z(R) := \{a \in R \mid a \cdot x = x \cdot a \ \ \forall\, x \in R\}$.

Theorem 7.5 (*Wedderburn*)

If R is a semisimple ring, then the following assertions hold.

(a) If $S \in \mathrm{Irr}(R)$, then $S(R^\circ) \neq 0$. Furthermore, $|\,\mathrm{Irr}(R)| < \infty$.

(b) We have
$$R^\circ = \bigoplus_{S \in \mathrm{Irr}(R)} S(R^\circ)\,,$$
where each homogenous component $S(R^\circ)$ is a two-sided ideal of R and $S(R^\circ)T(R^\circ) = 0$ if $S \neq T \in \mathrm{Irr}(R)$.

(c) Each $S(R^\circ)$ is a simple left Artinian ring, the identity element of which is an idempotent element of R lying in $Z(R)$.

Proof:

(a) By Exercise 7.3(e) every simple left R-module is isomorphic to a minimal left ideal of R, i.e. a simple submodule of R°. Hence if $S \in \operatorname{Irr}(R)$, then $S(R^\circ) \neq 0$. Now, by Theorem-Definition 6.2, the regular module admits a decomposition

$$R^\circ = \bigoplus_{j \in J} V_j$$

into a direct sum of a finite number of minimal left ideals V_j of R, and by Claim 1 in the proof of Proposition 7.1 any simple submodule of R° is isomorphic to V_j for some $j \in J$. Hence $|\operatorname{Irr}(R)| < \infty$.

(b) Proposition 7.1(iv) also yields $S(R^\circ) = \bigoplus_{V_j \cong S} V_j$ and Proposition 7.1(i) implies that

$$R^\circ = \bigoplus_{S \in \operatorname{Irr}(R)} S(R^\circ)\,.$$

Next notice that each homogeneous component is a left ideal of R, since it is by definition a sum of left ideals. Now let L be a minimal left ideal contained in $S(R^\circ)$, and let $x \in T(R^\circ)$ for a $T \in \operatorname{Irr}(R)$ with $S \neq T$. Then $Lx \subseteq T(R^\circ)$ and because $\varphi_x : R^\circ \longrightarrow R^\circ, m \mapsto mx$ is an R-endomorphism of R°, then either $Lx = \varphi_x(L)$ is zero or it is again a minimal left ideal, isomorphic to L. However, as $S \neq T$, we have $Lx = 0$. Therefore $S(R^\circ)T(R^\circ) = 0$, which implies that $S(R^\circ)$ is also a right ideal, hence two-sided.

(c) Part (b) implies that the homogeneous components are rings. Then, using Exercise 7.3(a), we may write $1_R = \sum_{S \in \operatorname{Irr}(R)} e_S$, where $S(R^\circ) = Re_S$ with e_S idempotent. Since $S(R^\circ)$ is a two-sided ideal, in fact $S(R^\circ) = Re_S = e_S R$. It follows that e_S is an identity element for $S(R^\circ)$.
To see that e_S is in the centre of R, consider an arbitrary element $a \in R$ and write $a = \sum_{T \in \operatorname{Irr}(R)} a_T$ with $a_T \in T(R^\circ)$. Since $S(R^\circ)T(R^\circ) = 0$ if $S \neq T \in \operatorname{Irr}(R)$, we have $e_S e_T = \delta_{ST}$. Thus, as e_T is an identity element for the T-homogeneous component, we have

$$\begin{aligned} e_S a &= e_S \sum_{T \in \operatorname{Irr}(R)} a_T = e_S \sum_{T \in \operatorname{Irr}(R)} e_T a_T = \sum_{T \in \operatorname{Irr}(R)} e_S e_T a_T \\ &= e_S a_S \\ &= a_S e_S \\ &= \sum_{T \in \operatorname{Irr}(R)} a_T e_T e_S = (\sum_{T \in \operatorname{Irr}(R)} a_T e_T) e_S = (\sum_{T \in \operatorname{Irr}(R)} a_T) e_S = a e_S\,. \end{aligned}$$

Finally, if $L \neq 0$ is a two-sided ideal in $S(R^\circ)$, then L must contain all the minimal left ideals of R isomorphic to S as a consequence of Exercise 7.3 (check it!). It follows that $L = S(R^\circ)$ and therefore $S(R^\circ)$ is a simple ring. It is left Artinian, because it is semissimple as an R-module. ■

Scholium 7.6

If R is a semisimple ring, then there exists a set of idempotent elements $\{e_S \mid S \in \operatorname{Irr}(R)\}$ such that

(i) $e_S \in Z(R)$ for each $S \in \operatorname{Irr}(R)$;

(ii) $e_S e_T = \delta_{ST} e_S$ for all $S, T \in \operatorname{Irr}(R)$;

(iii) $1_R = \sum_{S \in \operatorname{Irr}(R)} e_S$;

(iv) $R = \bigoplus_{S \in \operatorname{Irr}(R)} Re_S$, where each Re_S is a simple ring.

Idempotents satisfying Property (i) are called **central** idempotents, and idempotents satisfying Property (ii) are called **orthogonal**.

Remark 7.7

Remember that if R is a semisimple ring, then the regular module R° admits a composition series. Therefore it follows from the Jordan-Hölder Theorem that

$$R^\circ = \bigoplus_{S\in\mathrm{Irr}(R)} S(R^\circ) \cong \bigoplus_{S\in\mathrm{Irr}(R)} \bigoplus_{i=1}^{n_S} S$$

for uniquely determined integers $n_S \in \mathbb{Z}_{>0}$.

Theorem 7.8 (*Artin-Wedderburn*)

If R is a semisimple ring, then, as a ring,

$$R \cong \prod_{S\in\mathrm{Irr}(R)} M_{n_S}(D_S)\,,$$

where $D_S := \mathrm{End}_R(S)^{\mathrm{op}}$ is a division ring.

Before we proceed with the proof of the theorem, first recall that if we have a direct sum decomposition $U = U_1 \oplus \cdots \oplus U_r$ ($r \in \mathbb{Z}_{>0}$), then $\mathrm{End}_R(U)$ is isomorphic to the ring of $r \times r$-matrices in which the (i,j) entry lies in $\mathrm{Hom}_R(U_j, U_i)$. This is because any R-endomorphism $\phi : U \longrightarrow U$ may be written as a matrix of components $\phi = (\phi_{ij})_{1\leqslant i,j\leqslant r}$ where $\phi_{ij} : U_j \xrightarrow{inc.} U \xrightarrow{\phi} U \xrightarrow{proj.} U_i$, and when viewed in this way R-endomorphisms compose in the manner of matrix multiplication. (Known from the GDM-lecture if R is a field. The same holds over an arbitrary ring R.)

Proof: By Exercise 7.3(c), we have

$$\mathrm{End}_R(R^\circ) \cong R^{\mathrm{op}}$$

as rings. On the other hand, since $\mathrm{Hom}_R(S(R^\circ), T(R^\circ)) = 0$ for $S \ncong T$ (e.g. by Schur's Lemma, or by Proposition 7.1), the above observation yields

$$\mathrm{End}_R(R^\circ) \cong \prod_{S\in\mathrm{Irr}(R)} \mathrm{End}_R(S(R^\circ))$$

where $\mathrm{End}_R(S(R^\circ)) \cong M_{n_S}(\mathrm{End}_R(S)) \cong M_{n_S}(\mathrm{End}_R(S)^{\mathrm{op}})^{\mathrm{op}}$. Therefore, setting $D_S := \mathrm{End}_R(S)^{\mathrm{op}}$ yields the result. For by Schur's Lemma $\mathrm{End}_R(S)$ is a division ring, hence so is the opposite ring. ■

8 Semisimple algebras and their simple modules

From now on we leave the theory of modules over arbitrary rings and focus on finite-dimensional algebras over a field K. Algebras are in particular rings, and since K-algebras and their modules are in particular K-vector spaces, we may consider their dimensions to obtain further information. In particular, we immediately see that finite-dimensional K-algebras are necessarily left Artinian rings. Furthermore, the structure theorems of the previous section tell us that if A is a semisimple algebra over a field K, then

$$A^\circ = \bigoplus_{S\in\mathrm{Irr}(A)} S(A^\circ) \cong \bigoplus_{S\in\mathrm{Irr}(A)} \bigoplus_{i=1}^{n_S} S$$

where n_S corresponds to the multiplicity of the isomorphism class of the simple module S as a direct summand of A° in any given decomposition of A° into a finite direct sum of simple submodules. We shall see that over an algebraically closed field the number of simple A-modules is detected by the centre of A and also obtain information about the simple modules of algebras, which are not semisimple.

Exercise 8.1

Let A be an arbitrary K-algebra over a commutative ring K.

(a) Prove that $Z(A)$ is a K-subalgebra of A.

(b) Prove that if K is a field and $A \neq 0$, then $K \longrightarrow Z(A), \lambda \mapsto \lambda 1_A$ is an injective K-homomorphism.

(c) Prove that if $A = M_n(K)$, then $Z(A) = K\mathrm{I}_n$, i.e. the K-subalgebra of scalar matrices. (Hint: use the standard basis of $M_n(K)$.)

(d) Assume A is the algebra of 2×2 upper-triangular matrices over K. Prove that

$$Z(A) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} \right) \mid a \in K \right\} .$$

We obtain the following Corollary to Wedderburn's and Artin-Wedderburn's Theorems:

Theorem 8.2

Let A be a semisimple finite-dimensional algebra over an algebraically closed field K, and let $S \in \mathrm{Irr}(A)$ be a simple A-module. Then the following statements hold:

(a) $S(A^\circ) \cong M_{n_S}(K)$ and $\dim_K(S(A^\circ)) = n_S^2$;

(b) $\dim_K(S) = n_S$;

(c) $\dim_K(A) = \sum_{S \in \mathrm{Irr}(A)} \dim_K(S)^2$;

(d) $|\mathrm{Irr}(A)| = \dim_K(Z(A))$.

Proof:

(a) Since $K = \overline{K}$, Schur's Lemma implies that $\mathrm{End}_A(S) \cong K$. Hence the division ring D_S in the statement of the Artin-Wedderburn Theorem is $D_S = \mathrm{End}_A(S)^{\mathrm{op}} \cong K^{\mathrm{op}} = K$. Hence Artin-Wedderburn (and its proof) applied to the case $R = S(A^\circ)$ yields $S(A^\circ) \cong M_{n_S}(K)$. Hence $\dim_K(S(A^\circ)) = n_S^2$.

(b) Since $S(A^\circ)$ is a direct sum of n_S copies of S, (a) yields:

$$n_S^2 = n_S \cdot \dim_K(S) \quad \Longrightarrow \quad \dim_K(S) = n_S$$

(c) follows directly from (a) and (b).

(d) Since by Artin-Wedderburn and (a) we have $A \cong \prod_{S \in \mathrm{Irr}(A)} M_{n_S}(K)$, clearly

$$Z(A) \cong \prod_{S \in \mathrm{Irr}(A)} Z(M_{n_S}(K)) = \prod_{S \in \mathrm{Irr}(A)} K\mathrm{I}_{n_S} ,$$

where $\dim_K(K\mathrm{I}_{n_S}) = 1$. The claim follows. ∎

Corollary 8.3

Let A be a finite-dimensional algebra over an algebraically closed field K. Then the number of simple A-modules is equal to $\dim_K(Z(A/J(A)))$.

Proof: We have observed that A and $A/J(A)$ have the same simple modules (see Exercise 4.2), hence $|\operatorname{Irr}(A)| = |\operatorname{Irr}(A/J(A))|$. Moreover, the quotient $A/J(A)$ is J-semisimple by Proposition 6.7, hence semisimple by Proposition 6.6 because finite-dimensional algebras are left Artinian rings. Therefore it follows from Theorem 8.2(d) that

$$|\operatorname{Irr}(A)| = |\operatorname{Irr}(A/J(A))| = \dim_K\left(Z(A/J(A))\right).$$

■

Corollary 8.4

Let A be a finite-dimensional algebra over an algebraically closed field K. If A is commutative, then any simple A-module has K-dimension 1.

Proof: First assume that A is semisimple. As A is commutative, $A = Z(A)$. Hence parts (d) and (c) of Theorem 8.2 yield

$$|\operatorname{Irr}(A)| = \dim_K(A) = \sum_{S\in\operatorname{Irr}(A)} \underbrace{\dim_K(S)^2}_{\geqslant 1},$$

which forces $\dim_K(S) = 1$ for each $S \in \operatorname{Irr}(A)$.
Now, if A is not semissimple, then again we use the fact that A and $A/J(A)$ have the same simple modules (that is seen as abelian groups). Because $A/J(A)$ is semisimple and also commutative, the argument above tells us that all simple $A/J(A)$-modules have K-dimension 1. The claim follows. ■

Finally, we emphasise that in this section the assumption that the field K is algebraically closed is in general too strong and that it is possible to weaken this hypothesis so that Theorem 8.2, Corollary 8.3 and Corollary 8.4 still hold.

Indeed, if $K = \overline{K}$ is algebraically closed, then Part (b) of Schur's Lemma tells us that $\operatorname{End}_A(S) \cong K$ for any simple A-module S. This is the crux of the proof of Theorem 8.2. The following terminology describes this situation.

Definition 8.5

Let A be a finite-dimensional K-algebra. Then:

(a) A is called **split** if $\operatorname{End}_A(S) \cong K$ for every simple A-module S; and

(b) an extension field K' of K is called a **splitting field for** A if the K'-algebra $K' \otimes_K A$ is split.

Of course if A is split then K itself is a splitting field for A.

Remark 8.6

In fact for a finite-dimensional K-algebra A, the following assertions are equivalent:

(a) A is split;

(b) the product, for S running through $\operatorname{Irr}(A)$, of the structural homomorphisms $A \longrightarrow \operatorname{End}_K(S)$

(mapping $a \in A$ to the K-linear map $S \longrightarrow S$, $m \mapsto am$) induces an isomorphism of K-algebras

$$A/J(A) \cong \prod_{S \in \mathrm{Irr}(A)} \mathrm{End}_K(S)\,.$$

This is a variation of the Artin-Wedderburn Theorem we have seen in the previous section.