Chapter 11. Blocks

We can break down the representation theory of finite groups into its smallest parts: the *blocks* of the group algebra. First we will define blocks for arbitrary rings and then specify to the situation of a finite group G and a splitting p-modular system $(F, \mathcal{O}, k)$.

Notation: throughout this chapter, unless otherwise specified, we assume Assumption $(*)$ holds.

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37 The blocks of a ring

Throughout this section we let A denote an arbitrary associative ring with an identity element (denoted 1_A).

Proposition 37.1

(a) There is a bijection between

  (1) the set of decompositions

$$A = A_1 \oplus \cdots \oplus A_r \quad (r \in \mathbb{Z}_{\geqslant 1})$$

  of A into a direct sum of (A, A)-subbimodules (or equiv. of two-sided ideals of A), and

  (2) the set of decompositions

$$1_A = e_1 + \cdots + e_r$$

of 1_A into a sum of orthogonal idempotents of $Z(A)$,

in such a way that $e_i = 1_{A_i}$ and $A_i = Ae_i$ for every $1 \leqslant i \leqslant r$.

(b) For each $1 \leqslant i \leqslant r$, the direct summand A_i of A in (1) of (a) is indecomposable as an (A, A)-bimodule if and only if the corresponding central idempotent e_i is primitive.

(c) The decomposition of A into indecomposable (A, A)-subbimodules is uniquely determined.

Proof:

(a) Assume $A = A_1 \oplus \cdots \oplus A_r$ is a decomposition of A into (A, A)-bimodules. Then for each $1 \leqslant i \leqslant r$, there exists $e_i \in A_i$ such that
$$1_A = e_1 + \cdots + e_r\,.$$
Furthermore, for any element $a \in A$,
$$a = 1_A \cdot a = (e_1 + \cdots + e_r)a = e_1 a + \cdots + e_r a$$
and
$$a = a \cdot 1_A = a(e_1 + \cdots + e_r) = ae_1 + \cdots + ae_r\,.$$
So for each $1 \leqslant j \leqslant r$, we have $e_j a = ae_j \in A_j$, and it follows in particular that $e_1, \dots, e_r \in Z(A)$. Also, if $a_i \in A_i$ then $e_i a_i = a_i$ and $e_j a_i = 0$ if $j \neq i$. Hence, $e_i = 1_{A_i}$ and $e_i^2 = e_i$ so $\{e_i\}_{i=1}^r$ is a set of orthogonal idempotents of $Z(A)$, and $A_i = Ae_i$ for $1 \leqslant i \leqslant r$.

Conversely, if $\{e_i\}_{i=1}^r$ is a set of orthogonal idempotents of $Z(A)$ such that $1_A = \sum_{i=1}^r e_i$, then $A_i := Ae_i$ is an (A, A)-subbimodule of A for each $1 \leqslant i \leqslant r$ and $A_1 \oplus \cdots \oplus A_r$ is a direct sum decomposition of A into (A, A)-subbimodules.

(b) "$\Rightarrow$" Consider $A_i = Ae_i$ and suppose there exist orthogonal idempotents $f, h \in Z(A)$ such that $e_i = f + h$. Then $A_i = Af \oplus Ah$, where the sum is direct because if $a \in Af \cap Ah$ then $a = af$ and $a = ah$, hence $a = afh = a \cdot 0 = 0$. Thus, if e_i is not primitive, then A_i is decomposable as an (A, A)-bimodule.

"$\Leftarrow$" Now suppose that $A_i = Ae_i = L_1 \oplus L_2$, where L_1 and L_2 are two non-zero (A, A)-submodules. Then there exist $f \in L_1 \setminus \{0\}$ and $h \in L_2 \setminus \{0\}$ such that $e_i = f + h$. Now, $fh = 0$ since $fh \in L_1 \cap L_2 = \{0\}$. As e_i is the identity in A_i, we get
$$f = e_i f = (f + h)f = f^2 + hf = f^2 + 0 = f^2$$
and similarly, $h^2 = h$, hence f and h are orthogonal idempotents, so e_i is not primitive.

(c) Suppose that $A = A_1 \oplus \cdots \oplus A_r$ ($r \in \mathbb{Z}_{\geqslant 1}$) is a decomposition of A into indecomposable (A, A)-subbimodules, and suppose that L is an indecomposable direct summand of A from a different such decomposition of A. Every $x \in L$ can be written as
$$x = a_1 + \cdots + a_r \text{ with } a_i \in A_i \text{ for each } 1 \leqslant i \leqslant r,$$
so $L \ni e_i x = a_i$. Hence $L = (L \cap A_1) \oplus \cdots \oplus (L \cap A_r)$ and this is a decomposition of L. Since L is indecomposable there exists $1 \leqslant m \leqslant r$ such that $L = L \cap A_m$. By the indecomposability of A_m, this must be the whole of A_m. Thus the decomposition of A into (A, A)-subbimodules is uniquely determined. ■

Definition 37.2 (*Block, block idempotent, block algebra, belonging to a block*)

(a) The uniquely determined indecomposable (A, A)-bimodules $A_i = Ae_i$ $(1 \leqslant i \leqslant r)$ given by Theorem 37.1(c) are called the **blocks** of A and the corresponding primitive central idempotents e_i are called the associated **block idempotents**.
(The blocks are sometimes also called **block algebras** when A is an algebra.)

(b) We say that an (indecomposable) A-module M **belongs to (or lies in) the block** $A_i = Ae_i$ if $e_iM = M$ and $e_jM = 0$ for all $j \neq i$.

Exercise 37.3

Let $A = A_1 \oplus \cdots \oplus A_r$ $(r \in \mathbb{Z}_{\geqslant 1})$ be the block decomposition of A and let M be an arbitrary A-module. Prove that M admits a unique direct sum decomposition $M = M_1 \oplus \cdots \oplus M_r$ where for each $1 \leqslant i \leqslant r$ the summand M_i belongs to the block A_i of A. Deduce that every indecomposable A-module lies in a uniquely determined block of A.

Corollary 37.4

Let $A = A_1 \oplus \cdots \oplus A_r$ $(r \in \mathbb{Z}_{\geqslant 1})$ be the block decomposition of A and let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a s.e.s. of A-modules and A-module homomorphisms. Then, for each $1 \leqslant i \leqslant r$:

$$M \text{ lies in the block } A_i \text{ of } A \text{ if and only if } L \text{ and } N \text{ lie in } A_i\,.$$

In particular, if an A-module M lies in a block A_i of A, then so do all of its submodules and all of its factor modules.

Proof: Let $e_i \in Z(A)$ be the primitive idempotent corresponding to A_i. By Definition 37.2 an A-module belongs to the block $A_i = Ae_i$ if and only if external multiplication by e_i is an A-isomorphism on that module. Considering the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle e_i\cdot(-)} & & \downarrow{\scriptstyle e_i\cdot(-)} & & \downarrow{\scriptstyle e_i\cdot(-)} & & \downarrow{\scriptstyle\cong} \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

it follows from the five-Lemma that this property holds for M if and only if it holds for L and N. ■

38 p-blocks of finite groups

We now return to finite groups and observe the following.

Example 17 (*Blocks of FG*)

Since FG is semisimple, the block decomposition of FG is given by the Artin-Wedderburn Theorem. In particular, the blocks are matrix algebras and can be labelled by $\mathrm{Irr}(FG)$.

Remark 38.1 (*Blocks of $\mathcal{O}G$ and kG*)

The lifting of idempotents obtained in Theorem 32.3 implies that the quotient morphism $\mathcal{O}G \twoheadrightarrow [\mathcal{O}/\mathfrak{p}]G = kG, x \mapsto \overline{x} := x + \mathfrak{p}\cdot\mathcal{O}G$ induces a bijection

$$\begin{array}{ccc}\{\text{primitive idempotents of } Z(\mathcal{O}G)\} & \xleftrightarrow{\sim} & \{\text{primitive idempotents of } Z(kG)\} \\ e & \mapsto & \overline{e}\end{array}$$

Thus a decomposition $1_{\mathcal{O}G} = e_1 + \cdots + e_r$ of the identity element of $\mathcal{O}G$ into a sum of primitive central idempotents corresponds to a decomposition $1_{kG} = \overline{e}_1 + \cdots + \overline{e}_r$ of the identity element of kG into a sum of primitive central idempotents of kG. Therefore, by Proposition 37.1, there is a bijection between the blocks of $\mathcal{O}G$ and the blocks of kG mapping $B_i := e_i\mathcal{O}G$ to $\overline{B_i} := \overline{e_i}kG$.

Warning: The definition of a block of a finite group may vary from one book (resp. article) to another and different authors use different definitions of *blocks of finite groups* which can, from text to text, be *algebras*, *idempotents*, *sets of modules*, or *sets of ordinary characters and/or Brauer characters*. This may sound confusing at first sight, but in general, the context makes it clear which kind of objects is considered.

Definition 38.2 (*Blocks of finite groups, principal block*)

Let $K \in \{F, \mathcal{O}, k\}$. Then:

(a) The **blocks** of KG are the uniquely determined indecomposable (KG, KG)-bimodules defined by Theorem 37.1(c).

(b) We define a **p-block** of G to be the specification of a block of $\mathcal{O}G$, understanding also the corresponding block of kG, the corresponding idempotents of $\mathcal{O}G$ or of kG (or the modules which belongs to these). We write $\mathrm{Bl}_p(G)$ for the set of all p-blocks of G, resp. $\mathrm{Bl}_p(\mathcal{O}G)$ and $\mathrm{Bl}_p(kG)$ for the set of all blocks of $\mathcal{O}G$ and kG.

Exercise 38.3

Let $B \in \mathrm{Bl}_p(\mathcal{O}G)$. Prove that an $\mathcal{O}G$-module M belongs to B if and only if $M/\mathfrak{p}M$ belongs to the image $\overline{B} \in \mathrm{Bl}_p(kG)$ of B.

Example 18 (*p-blocks of a p-group*)

If G is a p-group, then G has a unique p-block. Indeed, we have already observed that in this case, the trivial module is the unique simple kG-module and hence kG has a unique PIM, namely kG itself. So kG is certainly also indecomposable as a (kG, kG)-bimodule. Therefore, $\mathrm{Bl}_p(kG) = \{kG\}$, and $\mathrm{Bl}_p(\mathcal{O}G) = \{\mathcal{O}G\}$.

Theorem 38.4

Let $S \not\cong T$ be simple kG-modules. Then the following assertions are equivalent:

(a) S and T belong to the same block of kG;

(b) there exist simple kG-modules $S = S_1, \ldots, S_m = T$ $(m \in \mathbb{Z}_{\geqslant 2})$ such that S_i and S_{i+1} are both composition factors of the same projective indecomposable kG-module for each $1 \leqslant i \leqslant m-1$.

Proof:

(b)⇒(a): This follows by induction on m because by Corollary 37.4 all the composition factors of a PIM of kG belong to the same block.

(a)⇒(b): Assume S belongs to the block B of kG. Decompose $B = P_1 \oplus \cdots \oplus P_s \oplus Q_1 \oplus \cdots \oplus Q_t$ in PIMs where composition factors of the P_i's are in relation (b) with S but none of the composition factors of the Q_j's are in relation (b) with S. To prove: $t = 0$. Clearly: then all simple modules T belonging to B are in relation (b) with S by Corollary 37.4 since its projective cover must be one of the P_i's. Now, by construction, no composition factor of the Q_j's can be a composition factor of the P_i's, so $\mathrm{Hom}_B(P_i, Q_j) = 0 = \mathrm{Hom}_B(Q_j, P_i)$ for every $1 \leqslant i \leqslant s$ and every $1 \leqslant j \leqslant r$. Thus both $P_1 \oplus \cdots \oplus P_s$ and $Q_1 \oplus \cdots \oplus Q_t$ are invariant under all B-endomorphisms, in particular under left and right multiplication by elements of B, so they must themselves be blocks of kG. The only possibility is $t = 0$. ■

The effect of this result is that the division of the simple kG-modules into blocks can be achieved in a purely combinatorial fashion, knowing the Cartan matrix of kG. The connection with a block matrix decomposition of the Cartan matrix is probably the origin of the use of the term *block* in representation theory.

Corollary 38.5

On listing the simple kG-modules so that modules in each block occur together, the Cartan matrix of kG has a block diagonal form, with one block matrix for each p-block of the group. Up to permutation of simple modules within p-blocks and permutation of the p-blocks, this is the unique decomposition of the Cartan matrix into block diagonal form with the maximum number of block matrices.

Proof: Given any matrix we may define an equivalence relation on the set of rows and columns of the matrix by requiring that a row be equivalent to a column if and only if the entry in that row and column is non-zero, and extending this by transitivity to an equivalence relation. In the case of the Cartan matrix the row indexed by a simple module S is in the same equivalence class as the column indexed by P_S, because S is a composition factor of P_S. If we order the rows and columns of the Cartan matrix so that the rows and columns in each equivalence class come together, the matrix is in block diagonal form, with square blocks, and this is the unique such expression with the maximal number of blocks (up to permutation of the blocks and permutation of rows and columns within a block). It follows Theorem 38.4 that the matrix blocks biject with the blocks of the group algebra. ■

39 Defect groups

To finish this chapter, for the sake of completeness, in this section and the next section we give an overview of important results concerning p-blocks, mostly with sketches of proofs only. We want to discuss the blocks of $\mathcal{O}G$ and kG, so we assume $K \in \{\mathcal{O}, k\}$.

Remark 39.1

Observe that any (KG, KG)-bimodule B becomes a left $K[G \times G]$-module in a natural way via the $G \times G$-action

$$\begin{aligned}(G \times G) \times B &\to B \\ \big((g,h), b\big) &\mapsto gbh^{-1}\end{aligned}$$

extended by K-bilinearity to the whole of $K[G \times G]$. This action is called the **conjugation action of** G on B. In particular, the group algebra KG itself is a left $K[G \times G]$-module and the blocks of KG can be viewed as indecomposable $K[G \times G]$-modules under this action.

Notation 39.2

Write $\Delta : G \longrightarrow G \times G, g \mapsto (g, g)$ for the diagonal embedding of G in $G \times G$.

Lemma 39.3

Regarded as an $K[G \times G]$-module KG is a transitive permutation module in which the stabiliser of 1_{KG} is $\Delta(G)$. Thus,

$$KG \cong K\uparrow_{\Delta(G)}^{G\times G}$$

as left $K[G \times G]$-module and any block of KG is $\Delta(G)$-projective.

Proof: By the definition of the action $G \times G$ permutes the K-basis of KG consisting of the group elements and

$$\mathrm{Stab}_{G\times G}(1) = \{(g_1, g_2) \in G \times G \mid g_1 \cdot 1 \cdot g_2^{-1} = 1\} = \Delta(G)\,.$$

Hence the first claim holds. The second claim now follows from Remark 30.2, and the last claim follows from the definition of relative projectivity since a block is by definition a direct summand of KG, which we see as $K[G \times G]$-module via Remark 39.1. ∎

Theorem 39.4

If $B \in \mathrm{Bl}_p(KG)$, then every vertex of B, considered as an indecomposable $K[G \times G]$-module, has the form $\Delta(D)$ for some p-subgroup $D \leqslant G$. Moreover, D is uniquely determined up to conjugation in G.

Proof: Since B is $\Delta(G)$-projective by Lemma 39.3 it has has a vertex contained in $\Delta(G)$, say Q, by definition 28.2. Moreover, by Proposition 28.4, Q is a p-subgroup, and thus there exists a p-subgroup $D \leqslant G$ such that $Q = \Delta(D)$, proving the first assertion.
Moreover, we know that $\Delta(D)$ is determined up to conjugation in $G \times G$ because it is a vertex of a module. Now, if $D_1 \leqslant G$ is another p-subgroup such that $\Delta(D_1)$ is a vertex of B, then there exists $(g_1, g_2) \in G \times G$ such that

$$\Delta(D_1) = {}^{(g_1,g_2)}\Delta(D)$$

and so for all $x \in D$, ${}^{(g_1,g_2)}(x, x) = ({}^{g_1}x, {}^{g_2}x) \in \Delta(D_1)$. Hence ${}^{g_1}x \in D_1$ for all $x \in D$. Finally, as $|D| = |D_1|$, it follows that ${}^{g_1}D = D_1$. ∎

Definition 39.5 (*Defect group, defect*)

Let $B \in \mathrm{Bl}_p(KG)$.

(a) A **defect group** of B is a p-subgroup D of G such that $\Delta(D)$ is a vertex of B considered as a left $K[G \times G]$-module.

(b) If the defect groups of B have order p^d ($d \in \mathbb{Z}_{\geqslant 0}$) then d is called the **defect** of B.

Note: As a defect group of a block is uniquely determined up to G-conjugacy, it is clear that in fact all defect groups have the same order.
Defect groups are useful and important because in some sense they measure how far a p-block is from being semisimple.

Theorem 39.6

Let $B \in \mathrm{Bl}_p(KG)$ and let D be a defect group of B. Then, every indecomposable KG-module belonging to B is relatively D-projective, and hence has a vertex contained in D.

Exercise 39.7

Prove Theorem 39.6.

[HINT: Prove that B, considered as KG-module via the conjugation action of G, is relatively D-projective.]

Corollary 39.8

Let B be a block of KG with a trivial defect group. Then B is a simple algebra, and in particular, is semisimple.

Proof: If B has a trivial defect group $D = \{1\}$, then by Theorem 39.6 every indecomposable kG-module belonging to B is $\{1\}$-projective, i.e. projective. So B is semisimple as all its modules are semisimple. But B is an indecomposable algebra by definition. Hence B is simple. ■

Main properties of the defect groups are the following.

Theorem 39.9 (*Green*)

Let B be a block of KG with defect group D. Then:

(a) if P is a Sylow p-subgroup of G containing D, then there exists $g \in C_G(D)$ such that $D = P \cap {}^gP$;

(b) D contains $O_p(G)$ and hence every normal p-subgroup of G;

(c) D is the largest normal p-subgroup of $N_G(D)$, i.e. $D = O_p(N_G(D))$.

Before, we prove the theorem, we recall that $O_p(G) = \bigcap_{P \in \mathrm{Syl}_p(G)} P$. Indeed, on the one hand, the intersection of the Sylow p-subgroups is a normal p-subgroup since $\mathrm{Syl}_p(G)$ is closed under conjugation, proving that $O_p(G) \supseteq \bigcap_{P \in \mathrm{Syl}_p(G)} P$. On the other hand $O_p(G) \subseteq P$ for some Sylow p-subgroup P, and hence ${}^g(O_p(G)) = O_p(G) \subseteq {}^gP$ for every element $g \in G$. Since $\mathrm{Syl}_p(G) = \{{}^gP \mid g \in G\}$, by Sylow's theorems, it follows that $O_p(G)$ is contained in their intersection.

Proof: (a) Lemma 39.3 and the Mackey formula yield

$$\begin{aligned} KG\downarrow^{G\times G}_{P\times P} &= K\uparrow^{G\times G}_{\Delta(G)}\downarrow^{G\times G}_{P\times P} \\ &= \bigoplus_{x\in[P\times P\backslash G\times G/\Delta(G)]} {}^xK\downarrow^{{}^x\Delta(G)}_{(P\times P)\cap {}^x\Delta(G)}\uparrow^{P\times P}_{(P\times P)\cap {}^x\Delta(G)} \\ &= \bigoplus_{x\in[P\times P\backslash G\times G/\Delta(G)]} K\uparrow^{P\times P}_{(P\times P)\cap {}^x\Delta(G)} \end{aligned}$$

where each summand $K\uparrow^{P\times P}_{(P\times P)\cap {}^x\Delta(G)}$ is indecomposable by Example 14 as $P \times P$ is a p-group. Now, $B\downarrow^{G\times G}_{P\times P}$ has an indecomposable direct summand M such that $M\downarrow^{P\times P}_{\Delta(D)}$ has a $\Delta(D)$-source of B as a summand. Then $\Delta(D)$ is also a vertex of M, and M must have the form $K\uparrow^{P\times P}_{(P\times P)\cap {}^x\Delta(G)}$ for some $x \in G \times G$. Thus $\Delta(D)$ is $P \times P$-conjugate to $(P \times P) \cap {}^x\Delta(G)$, i.e. there exists $z \in P \times P$ such that

$$\Delta(D) = {}^z((P \times P) \cap {}^x\Delta(G))\,.$$

Observe now that the subset $1 \times G = \{(1,t) \in G \times G \mid t \in G\}$ forms a set of coset representatives for $\Delta(G)$ in $G \times G$, and so we may assume that $x = (1,t)$ for some $t \in G$. Also, we may write $z = (r,s)$ where $r, s \in P$. With this notation we compute that

$$\begin{aligned}(P \times P) \cap {}^{x}\Delta(G) &= (P \times P) \cap {}^{(1,t)}\Delta(G) \\ &= \{(x, {}^{t}x) \in G \times G \mid x \in P \text{ and } {}^{t}x \in P\} \\ &= \{(x, {}^{t}x) \in G \times G \mid x \in P \cap {}^{t^{-1}}P\} \\ &= {}^{(1,t)}\Delta(P \cap {}^{t^{-1}}P)\end{aligned}$$

and so

$$\Delta(D) = {}^{z}((P \times P) \cap {}^{x}\Delta(G)) = {}^{(r,s)(1,t)}\Delta(P \cap {}^{t^{-1}}P)\,.$$

Now, as $r \in P$, projecting onto the first coordinate yields

$$\begin{aligned}D &= {}^{r}(P \cap {}^{t^{-1}}P) \\ &= {}^{r}P \cap {}^{rt^{-1}}P \\ &= P \cap {}^{rt^{-1}}P\,.\end{aligned}$$

It may, however, be that rt^{-1} does not centralise D. If not, then observe that

$${}^{(1,t^{-1})(r^{-1},s^{-1})}\Delta(D) = \Delta(P \cap {}^{t^{-1}}P) \subseteq \Delta(G)\,,$$

so that ${}^{r^{-1}}y = {}^{t^{-1}s^{-1}}y$ for all $y \in D$. Hence $rt^{-1}s^{-1} \in C_G(D)$ and as $s \in P$ we have $D = P \cap {}^{rt^{-1}s^{-1}}P$, proving the assertion.

(b) Clearly $D \supseteq O_p(G)$ since by (a) D is the intersection of two Sylow p-subgroups and hence contains $\bigcap_{P \in \mathrm{Syl}_p(G)} P = O_p(G)$.

(c) Let $P \in \mathrm{Syl}_p(G)$ be such that it contains a Sylow p-subgroup of $N_G(D)$. Then, such a P satisfies $P \cap N_G(D) \in \mathrm{Syl}_p(N_G(D))$. Now, by (a), there exists $g \in C_G(D)$ such that $D = P \cap {}^{g}P$ and so in particular $g \in N_G(D)$. Thus ${}^{g}P \cap N_G(D) = {}^{g}(P \cap N_G(D))$ is also a Sylow p-subgroup of $N_G(D)$, and

$$D = (P \cap N_G(D)) \cap {}^{g}(P \cap N_G(D))$$

is the intersection of two Sylow p-subgroups of $N_G(D)$. Thus $D \supseteq O_p(N_G(D))$. But on the other hand D is a normal p-subgroup of $N_G(D)$, and so is contained in $O_p(N_G(D))$, proving that equality holds, as required. ■

40 Brauer's 1st and 2nd Main Theorems

We continue our discussion of the blocks of $\mathcal{O}G$ and kG, and so we assume that $K \in \{\mathcal{O}, k\}$. We present here two main results due to Brauer and related concepts.

Definition 40.1

Let $H \leqslant G$, let $b \in \mathrm{Bl}_p(KH)$. Then a block $B \in \mathrm{Bl}_p(KG)$ **corresponds to** b if and only if $b \mid B{\downarrow}^{G\times G}_{H\times H}$, *and* B is the unique block of KG with this property. We then write $B = b^G$. If such a block B exists, then we say that b^G **is defined**.

Remark 40.2

Let $H \leqslant G$, and let $Q \leqslant H$ be a p-subgroup such that $C_G(Q) \leqslant H$. Note that

$$KG\downarrow^{G\times G}_{H\times H} = \bigoplus_{t\in[H\backslash G/H]} KHtH \ = KH \oplus \bigoplus_{t\in[H\backslash G/H], t\notin H} KHtH.$$

Fact: if $t \notin H$ then the $K[H\times H]$-submodule $KHtH$ of KG has no direct summands with vertex containing $\Delta(Q)$.

In particular, if X is an indecomposable direct summand of $KG\downarrow^{G\times G}_{H\times H}$ with vertex containing $\Delta(Q)$, then X is a direct summand of KH, so X is a block of KH.

Proposition 40.3 (*Facts about b^G*)

Let $H \leqslant G$ and let b be a block of KH with defect group D.

(a) If b^G is defined, then D lies in a defect group of b^G.

(b) If $H \leqslant N \leqslant G$, and b^N, $(b^N)^G$ and b^G are defined, then $b^G = (b^N)^G$.

(c) If $C_G(D) \leqslant H$ then b^G is defined.

Exercise 40.4

Prove Proposition 40.3.

[Hints for (a): Let E be a defect group of $B := b^G$. Then B is a direct summand of $V\uparrow^{G\times G}_{\Delta(E)}$ for some $\Delta(E)$-module V. Consider $V\uparrow^{G\times G}_{\Delta(E)}\downarrow^{G\times G}_{H\times H}$.

Hints for (b): Part (b) essentially follows from the definitions.

Hints for (c): Justify that it is enough to prove that b occurs precisely once in a decomposition of $KG\downarrow^{G\times G}_{H\times H}$ into indecomposable modules. Use the remark before the proposition.]

Theorem 40.5 (*Brauer's First Main Theorem*)

Let $D \leqslant G$ be a p-subgroup and let $H \leqslant G$ containing $N_G(D)$. Then there is a bijection

$$\{\text{Blocks of } KH \text{ with defect group } D\} \xrightarrow{\sim} \{\text{Blocks of } KG \text{ with defect group } D\}$$
$$b \mapsto b^G$$

Moreover, in this case b^G is called the **Brauer correspondent** of b.

Proof (Sketch) : Regarding the blocks of KH as left $K[H\times H]$-modules and the blocks of KG as left $K[G\times G]$-modules, the given bijection is a particular case of the Green correspondence. ∎

Definition 40.6 (*Brauer correspondence*)

The bijection in Brauer's First Main Theorem is called the **Brauer correspondence**.

Exercise 40.7

Verify that the Brauer correspondence is a particular case of the Green correspondence.

Theorem 40.8 (*Brauer's Second Main Theorem*)

Let $H \leqslant G$, let $B \in \mathrm{Bl}_p(KG)$ and let $b \in \mathrm{Bl}_p(KH)$. Suppose that V is an indecomposable module belonging to B and U is an indecomposable module belonging to b with vertex Q such that $C_G(Q) \leqslant H$. If U is a direct summand of $V{\downarrow}^G_H$, then b^G is defined and $b^G = B$.

Lemma 40.9

Let S be a simple kG-module. Then $O_p(G)$ acts trivially on S. In particular, the simple kG-modules are precisely the simple $k[G/O_p(G)]$-modules inflated to kG, i.e.

$$\begin{aligned}\mathrm{Inf}^G_{G/O_p(G)}:\ \mathrm{Irr}(k[G/O_p(G)]) &\xrightarrow{\sim} \mathrm{Irr}(kG)\\ T &\mapsto \mathrm{Inf}^G_{G/O_p(G)}(T)\end{aligned}$$

is a bijection.

Proof: Since $O_p(G) \trianglelefteq G$, we know from Clifford's Theorem that $S{\downarrow}^G_{O_p(G)}$ is semisimple, hence of the form

$$S{\downarrow}^G_{O_p(G)} = k \oplus \cdots \oplus k$$

since $O_p(G)$ is a p-group and hence has only one simple module up to isomorphism, namely the trivial module. In other words, $O_p(G)$ acts trivially on S. The second claim follows immediately. ■

Corollary 40.10

Let B be a block of kG with a defect group D. Then there exists an indecomposable kG-module belonging to B with vertex D.

Proof: Write $N := N_G(D)$. Let $b \in \mathrm{Bl}_p(kN)$ be the Brauer correspondent of b. As D is a defect group of b, $D = O_p(N)$ by Theorem 39.9(c). Now, let S be a simple kN-module belonging to b. Then, by Lemma 40.9, D acts trivially on S. So S can be viewed as a simple $k[N/D]$-module and we let P_S be the projective cover of S seen as a $k[N/D]$-module. Then, by Corollary 37.4, the inflation $\mathrm{Inf}^N_{N/D}(P_S)$ of P_S to N is an indecomposable kN-module belonging to b.

Claim 1: D is a vertex of $\mathrm{Inf}^N_{N/D}(P_S)$.
Indeed: By definition $P_S \mid k[N/D]$, so

$$\mathrm{Inf}^N_{N/D}(P_S) \mid \mathrm{Inf}^N_{N/D}(k[N/D]) = \mathrm{Inf}^N_{N/D}(\mathrm{Ind}^{N/D}_{D/D}(k)) = \mathrm{Ind}^N_D(\mathrm{Inf}^D_{D/D}(k)) = k{\uparrow}^N_D$$

and is therefore D-projective. Now, since $D \trianglelefteq N$, it follows from Clifford's Theorem that $k{\uparrow}^N_D{\downarrow}^N_D$ is a direct sum of N-conjugates of the trivial kD-module k, which are all again trivial since D is a p-group. So

$$\mathrm{Inf}^N_{N/D}(P_S){\downarrow}^N_D \mid k \oplus \cdots \oplus k\,.$$

Therefore the indecomposable direct summands of $\mathrm{Inf}^N_{N/D}(P_S){\downarrow}^N_D$ all have vertex D. It follows then from Lemma 29.1 that D is a vertex of $\mathrm{Inf}^N_{N/D}(P_S)$, because $\mathrm{Inf}^N_{N/D}(P_S){\downarrow}^N_D$ has at least one indecomposable direct summand with the same vertex as $\mathrm{Inf}^N_{N/D}(P_S)$. This proves Claim 1.

Next we observe that the kG-Green correspondent $f(\mathrm{Inf}^N_{N/D}(P_S))$ of $\mathrm{Inf}^N_{N/D}(P_S)$ is certainly an indecomposable kG-module with vertex D.

Claim 2: $f(\mathrm{Inf}^N_{N/D}(P_S))$ belongs to B.
Indeed, this is clear by definition of the Green correspondent and Brauer's First Main Theorem.

■

The next result shows that the converse of Corollary 39.8 also holds.

Corollary 40.11

A block B of kG is a simple algebra if and only if B has a trivial defect group.

Proof: If $B \in \mathrm{Bl}_p(kG)$ has trivial defect group, then B is a simple algebra by Corollary 39.8. Suppose now that B is a block of kG which is a simple algebra. Then B is semisimple so all B-modules are projective. Hence all indecomposable B-modules have trivial vertices so, by Corollary 40.10, B has a trivial defect group. ■

Definition 40.12 (*Principal block*)

(a) The **principal block** of kG is the block of kG to which the trivial kG-module k belongs. This block is denote by $B_0(kG)$.

(b) A block of kG whose defect groups are the Sylow p-subgroups of G is said to have (or to be of) **full defect**.

Lemma 40.13

(a) Any indecomposable kG-module whose vertices are the Sylow p-subgroups of G belongs to a block of kG of full defect.

(b) The principal block $B_0(kG)$ is a block of full defect.

Proof: (a) is clear by Theorem 39.6.

(b) is clear by (a) since the vertices of the trivial module are the Sylow p-subgroups of G. ■