Chapter 10. Brauer Characters

Recall that if F is a field of characteristic zero, then FG-modules are isomorphic if and only if their characters are equal. Also, the character of an FG-module provides complete information about its composition factors, including multiplicities, provided that the irreducible characters are known. All this does not hold for fields k of characteristic $p > 0$. For instance, if W is a k-vector space on which G acts trivially and $\dim_k(W) = ap + 1$ for some non-negative integer a, then the k-character of W is the trivial character. This implies that a k-character can only give information about multiplicities of composition factors modulo p. In view of these issues, the aim of this chapter is to define a slightly different kind of *character theory* for modular representations of finite groups and to establish links with ordinary character theory.

Notation: Throughout, G denotes a finite group and p a prime number. We let $(F, \mathcal{O}, k)$ denote a p-modular system and we assume F contains all $\exp(G)$-th roots of unity, so $(F, \mathcal{O}, k)$ is a splitting p-modular system for G and all its subgroups (see Theorem 14.2). We write $\mathfrak{p} := J(\mathcal{O})$. For $K \in \{F, \mathcal{O}, k\}$ all KG-modules considered are assumed to be free of finite rank over K. If $K \in \{F, k\}$ and U is a KG-module, then we write $\rho_U : G \longrightarrow \mathrm{GL}(U)$ for the underlying K-representation.

For background results in ordinary character theory I refer to my Skript *Character Theory of Finite Groups* from the SS 2020 / SS 2022.

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Before we start, we recall the following useful terminology from finite group theory.

Definition

Let G be a finite group and p be a prime number.

(a) If $|G| = p^a m$ with $a, m \in \mathbb{Z}_{\geqslant 0}$ and $(p, m) = 1$, then $|G|_p := p^a$ is the p-**part** of the order $|G|$ of G and $|G|_{p'} := m$ is the p'-**part** of the order of G.

(b) An element $g \in G$ is called a p-**regular element** (or a p'-**element**) if $p \nmid o(g)$ and we write

$$G_{p'} := \{g \in G \mid p \nmid o(g)\}$$

for the set of all p-regular elements of G. (Warning: in general this is not a subgroup!)

(c) An element $g \in G$ is called a p-**singular element** if $p \mid o(g)$ and it is called a p-**element** if $o(g)$ is a power of p.

Remark

Let G be a finite group and p be a prime number. Let $g \in G$ and write $o(g) = p^n m$ with $n, m \in \mathbb{Z}_{\geqslant 0}$ such that $(p, m) = 1$. Then, letting $a, b \in \mathbb{Z}$ be such that $1 = ap^n + bm$, we may set

$$g_p := g^{bm} \quad \text{and} \quad g_{p'} := g^{ap^n}$$

It is immediate that

$$(g_p)^{p^n} = 1, \quad (g_{p'})^m = 1, \quad \text{and} \quad g = g_p g_{p'}\,.$$

Thus, g_p is a p-element and is called the p-**part** of g, and $g_{p'}$ is p-regular and is called the p'-**part** of g.

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Since we assume that the given p-modular system $(F, \mathcal{O}, k)$ is such that F contains all $\exp(G)$-th roots of unity, both F and k contain a primitive a-th root of unity, where a is the l.c.m. of the orders of the p-regular elements of G. To start with we examine the relationship between the roots of unity in F and in k. Set

$$\mu_F := \{a\text{-th roots of 1 in } F\}\,;$$

$$\mu_k := \{a\text{-th roots of 1 in } k\}\,.$$

Then $\mu_F \subseteq \mathcal{O}$ (since roots of unity take value 1 under the valuation) and, as both μ_F and μ_k are finite groups, it follows from Corollary 13.8 that the quotient morphism $\mathcal{O} \twoheadrightarrow \mathcal{O}/\mathfrak{p}$ restricted to μ_F induces a group isomorphism

$$\mu_F \xrightarrow{\cong} \mu_k\,.$$

We write the underlying bijection as $\widehat{\xi} \mapsto \xi$, so that if ξ is an a-th root of unity in k then $\widehat{\xi}$ is the unique a-th root of unity in $\mathcal{O}$ which maps onto it.

Lemma 35.1 (*Diagonalisation lemma*)

Let $\rho : G \longrightarrow \mathrm{GL}(U)$ be a k-representation of G. Then, for every p-regular element $g \in G_{p'}$, the k-linear map $\rho(g)$ is diagonalisable and its eigenvalues are $o(g)$-th roots of unity lying in μ_k. In other words, there exists an ordered k-basis B of U with respect to which

$$\big(\rho(g)\big)_B = \begin{bmatrix} \xi_1 & 0 & \cdots & \cdots & 0 \\ 0 & \xi_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \xi_n \end{bmatrix},$$

where $n := \dim_k(U)$ and each ξ_i ($1 \leqslant i \leqslant n$) is an $o(g)$-th root of unity in k.

Proof: Let $g \in G_{p'}$. It is enough to consider the restriction of ρ to the cyclic subgroup $\langle g \rangle$. Since $p \nmid |\langle g \rangle|$, $k\langle g \rangle$ is semisimple by Maschke's Theorem. Moreover, as k is a splitting field for $\langle g \rangle$, it follows from Corollary 12.3 that all irreducible k-representations of $\langle g \rangle$ have degree 1. Hence $\rho|_{\langle g \rangle}$ can be decomposed as the direct sum of degree 1 subrepresentations. As a consequence $\rho(g) = \rho|_{\langle g \rangle}(g)$ is diagonalisable and there exists a k-basis B of U satisfying the statement of the lemma. It follows immediately that the eigenvalues are $o(g)$-th roots of unity because $\rho_U(g^{o(g)}) = \rho_U(1_G) = \mathrm{Id}_U$. They all lie in μ_k, being $o(g)$-th roots of unity, hence a-th roots of unity. ■

This leads to the following definition.

Definition 35.2 (*Brauer characters*)

Let U be a kG-module of dimension $n \in \mathbb{Z}_{\geqslant 0}$ and let $\rho_U : G \to \mathrm{GL}(U)$ be the associated k-representation. The **p-Brauer character** or simply the **Brauer character** of G afforded by U (resp. of ρ_U) is the F-valued function

$$\begin{aligned} \varphi_U : G_{p'} &\to \mathcal{O} \subseteq F \\ g &\mapsto \widehat{\xi_1} + \cdots + \widehat{\xi_n}, \end{aligned}$$

where $\xi_1, \ldots, \xi_n \in \mu_k$ are the eigenvalues of $\rho_U(g)$. The integer n is also called the **degree** of φ_U. Moreover, φ_U is called **irreducible** if U is simple (resp. if ρ_U is irreducible), and it is called **linear** if $n = 1$. We let $\mathrm{IBr}_p(G) := \{\varphi_S \mid S \in \mathrm{Irr}(kG)\}$ be the set of all irreducible Brauer characters of G.

In the sequel, we want to prove that Brauer characters of kG-modules have properties similar to $\mathbb{C}$-characters.

Remark 35.3

(a) **Warning**: $\varphi(g) \in \mathcal{O} \subseteq F$ even though $\rho_U(g)$ is defined over the field k of characteristic $p > 0$.

(b) Often the values of Brauer characters are considered as complex numbers, i.e. sums of complex roots of unity. Of course, in that case then $\varphi_U(g)$ depends on the choice of embedding of μ_F into $\mathbb{C}$. However, for a fixed embedding, $\varphi_U(g)$ is uniquely determined up to similarity of $\rho_U(g)$.

Immediate properties of Brauer characters are as follows.

Proposition 35.4

Let U be a kG-module with Brauer character φ_U. Then:

(a) $\varphi_U(1) = \dim_k(U)$;

(b) φ_U is a class function on $G_{p'}$;

(c) $\varphi_U(g^{-1}) = \varphi_{U^*}(g)\ \ \forall\, g \in G_{p'}$.

Proof: Let $n := \dim_k(U)$ and let ρ_U be the k-representation associated to U.

(a) Clearly, $\rho_U(1_G) = \mathrm{Id}_U$ and has n eigenvalues all equal to 1_k. Since $\widehat{1}_k = 1_F$ the sum of the lifts of these n eigenvalues is $n = \dim_k(U)$.

(b) If $g, x \in G_{p'}$, then $\rho_U(xgx^{-1}) = \rho_U(x)\rho_U(g)\rho_U(x)^{-1}$, so $\rho_U(g)$ and $\rho_U(xgx^{-1})$ are similar and therefore have the same eigenvalues. Thus, by definition of φ_U we have $\varphi_U(g) = \varphi_U(xgx^{-1})$. [Note that in this proof we could take $x \in G$ and it would not change the result!]

(c) Let $g \in G_{p'}$ and let $\xi_1, \ldots, \xi_n \in \mu_k$ be the eigenvalues of $\rho_U(g)$. As $\rho_U(g)$ is diagonalisable by Lemma 35.1, it follows immediately that the eigenvalues of $\rho_U(g^{-1}) = \rho_U(g)^{-1}$ are $\xi_1^{-1}, \ldots, \xi_n^{-1}$, as are the eigenvalues of $\rho_{U^*}(g) = \rho_U(g^{-1})^{tr}$. Since $\mu_F \longrightarrow \mu_k, \widehat{\xi} \mapsto \xi$ is a group isomorphism, we have $\widehat{\xi_i^{-1}} = \widehat{\xi_i}^{-1}$ for each $1 \leqslant i \leqslant n$, and the claim follows. ■

Exercise 35.5

Let U, V, W be kG-modules. Prove the following assertions.

(a) If $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is a s.e.s. of kG-modules, then $\varphi_V = \varphi_U + \varphi_W$.

(b) If $U \neq 0$ and the pairwise non-isomorphic composition factors of U are $S_1, \ldots, S_m$ ($m \in \mathbb{Z}_{\geqslant 1}$) with multiplicities $n_1, \ldots, n_m$ respectively, then

$$\varphi_U = n_1\varphi_{S_1} + \ldots + n_m\varphi_{S_m}.$$

In particular, if two kG-modules have isomorphic composition factors, counting multiplicities, then they have the same Brauer character.

(c) We have $\varphi_{U\oplus V} = \varphi_U + \varphi_V$ and $\varphi_{U\otimes_k V} = \varphi_U \cdot \varphi_V$.

Exercise 35.6

Exercise 35.5(a) tells us that two isomorphic kG-modules afford the same Brauer character. However, the converse does not hold as Exercise 35.5(b) tells us that two kG-modules with the same composition factors afford the same Brauer character. Consider for example the group $G = C_3$ and assume that $p = 3$. Then, we know from Theorem 26.1 that there exists a unique kG-module M such that $\dim_k(M) = 2$. Moreover, as G is a 3-group, it has a unique simple module, up to isomorphism, namely the trivial module k. It follows that M has two composition factors, both equal to k and we have

$$\varphi_M = \varphi_k + \varphi_k = \varphi_{k\oplus k}.$$

The link between the Brauer characters and the usual notion of a character, i.e. trace functions, is described below and explains why trace functions are not *good enough*.

Definition 35.7 (*character*)

Assume $K \in \{F, k\}$. Let U be a KG-module and let $\rho_U : G \longrightarrow \mathrm{GL}(U)$ be the underlying K-representation, then the trace function

$$\begin{array}{rccl} \mathrm{Tr}(-,U): & G & \longrightarrow & K \\ & g & \mapsto & \mathrm{Tr}(g,U) := \mathrm{Tr}\left(\rho_U(g)\right) \end{array}$$

is called the K**-character** of U.

For $K = F$, we also write χ_U instead of $\mathrm{Tr}(-,U)$ and we write $\mathrm{Irr}_F(G)$ for the set of all irreducible F-characters of G.

(Recall from *Character Theory of Finite Groups* that an F-character χ_U is called irreducible if U is a simple FG-module.)

Lemma 35.8

Let $U \neq 0$ be a kG-module and let $g \in G$. Then:

(a) $\mathrm{Tr}(g,U) = \mathrm{Tr}(g_{p'},U)$, and

(b) $\mathrm{Tr}(g,U) = \varphi_U(g_{p'}) + \mathfrak{p}$ in the quotient $k = \mathcal{O}/\mathfrak{p}$.

Proof: Let ρ_U be the k-representation associated to U.

(a) Clearly $\rho_U(g) = \rho_U(g_p)\rho_U(g_{p'}) = \rho_U(g_{p'})\rho_U(g_p)$. Therefore, we know from linear algebra that the eigenvalues of $\rho_U(g)$ are pairwise products of suitably ordered eigenvalues of $\rho_U(g_{p'})$ and $\rho_U(g_p)$. Now, as g_p is a p-element, a p-power of $\rho_U(g_p)$ is the identity and so all the eigenvalues of $\rho_U(g_p)$ are equal to one because $\mathrm{char}(k) = p$. Hence $\rho_U(g)$ and $\rho_U(g_{p'})$ have the same eigenvalues (counted with multiplicities) and the claim follows.

(b) Let $\xi_1, \ldots, \xi_n$ be the eigenvalues of $\rho_U(g_{p'})$. Then, by definition, $\varphi_U(g_{p'}) = \hat{\xi}_1 + \cdots + \hat{\xi}_n$, where, for each $1 \leqslant i \leqslant n$, $\hat{\xi}_i$ is such that $\hat{\xi}_i + \mathfrak{p} = \xi_i$ in the quotient $k = \mathcal{O}/\mathfrak{p}$. Therefore, it follows from (a) and the diagonalisation lemma that

$$\mathrm{Tr}(g,U) = \mathrm{Tr}(g_{p'},U) = \xi_1 + \cdots + \xi_n = (\hat{\xi}_1 + \mathfrak{p}) + \cdots + (\hat{\xi}_n + \mathfrak{p}) = (\hat{\xi}_1 + \ldots + \hat{\xi}_n) + \mathfrak{p} = \varphi_U(g_{p'}) + \mathfrak{p}\,.$$ ■

Notation 35.9

We let $\mathrm{Cl}_F(G) := \{f : G \longrightarrow F \mid f \text{ class function}\}$ denote the F-vector space of F-valued class functions on G and we let $\mathrm{Cl}_F(G_{p'}) := \{f : G_{p'} \longrightarrow F \mid f \text{ class function}\}$ denote the F-vector space of F-valued class functions on $G_{p'}$.

Proposition 35.10

A class function $\varphi \in \mathrm{Cl}_F(G_{p'})$ is a Brauer character if and only if it is a non-negative $\mathbb{Z}$-linear combination of elements of $\mathrm{IBr}_p(G)$.

Proof: The necessary condition is clear by Exercise 35.5(b). Conversely, let $n_1, \ldots, n_m \in \mathbb{Z}_{\geqslant 0}$ and let $\varphi_{S_1}, \ldots, \varphi_{S_m} \in \mathrm{IBr}_p(G)$ ($m \in \mathbb{Z}_{\geqslant 0}$) be the Brauer characters afforded by the simple kG-modules $S_1, \ldots, S_m$. Then, by Exercise 35.5(c), the class function $\varphi := n_1\varphi_{S_1} + \ldots + n_m\varphi_{S_m}$ is the Brauer character afforded by the kG-module

$$S_1^{n_1} \oplus \ldots \oplus S_m^{n_m}\,.$$ ■

Theorem 35.11

The set $\mathrm{IBr}_p(G)$ of irreducible Brauer characters of G is F-linearly independent and hence

$$|\,\mathrm{IBr}_p(G)| \leqslant \dim_F \mathrm{Cl}_F(G_{p'}) = \text{number of conjugacy classes of } p\text{-regular elements in } G\,.$$

In particular, $\mathrm{IBr}_p(G) = \{\varphi_k\}$ provided G is a finite p-group.

Without proof.
The second claim is obvious, because the indicator functions on the conjugacy classes of p-regular elements form an F-basis. The third claim is clear since if G is a p-group, then $G_{p'} = \{1_G\}$. Thus $|\,\mathrm{IBr}_p(G)| = 1$ by the first part, namely $\mathrm{IBr}_p(G) = \{\varphi_k\}$.

36 Back to decomposition matrices of finite groups

We now want to investigate the connections between representations of G over F (or $\mathbb{C}$) and representations of G over k through the connections between their F-characters and Brauer characters.

Lemma 36.1

Let V be an FG-module with F-character χ_V. Let L be an $\mathcal{O}$-form of V and let $\overline{L} = L/\mathfrak{p}L$ be the reduction modulo $\mathfrak{p}$ of L. Then,

$$\chi_V|_{G_{p'}} = \varphi_{\overline{L}}\,,$$

and is often called the **reduction modulo** p **of** χ_V.

Proof: Let $g \in G_{p'}$ and write $R_L : G \longrightarrow \mathrm{GL}(L)$ for the $\mathcal{O}$-representation associated to L. Since g is p-regular, the eigenvalues $x_1, \dots, x_n$ of $R_L(g)$ belong to $\mu_F \subseteq \mathcal{O}$ and $\overline{x_1}, \dots, \overline{x_n} \in k$ are the eigenvalues of $\rho_{\overline{L}}(g)$. Since by definition $\varphi_{\overline{L}}$ is the sum of the lifts of the latter quantities, we have

$$\varphi_{\overline{L}}(g) = x_1 + \ldots + x_n = \mathrm{Tr}(g, L^F) = \mathrm{Tr}(g, V) = \chi_V(g)$$

since L is an $\mathcal{O}$-form of V. ■

Proposition 36.2

(a) The set $\mathrm{IBr}_p(G)$ is an F-basis of $\mathrm{Cl}_F(G_{p'})$.

(b) Given an irreducible F-character $\chi \in \mathrm{Irr}_F(G)$, there exist non-negative integers $d_{\chi\varphi}$ such that

$$\chi|_{G_{p'}} = \sum_{\varphi \in \mathrm{IBr}_p(G)} d_{\chi\varphi}\varphi.$$

(c) $|\,\mathrm{IBr}_p(G)| = $ number of p-regular conjugacy classes of G.

Proof: By Theorem 35.11 it only remains to prove that $\mathrm{IBr}_p(G)$ spans $\mathrm{Cl}_F(G_{p'})$. So let $\mu \in \mathrm{Cl}_F(G_{p'})$ be an arbitrary F-valued class function on $G_{p'}$. Then μ can be extended to $\mu^{\#} \in \mathrm{Cl}_F(G)$, e.g. by setting

$\mu^{\#}(g) = 0$ for every $g \in G \backslash G_{p'}$. Since $\mathrm{Irr}_F(G)$ is an F-basis for $\mathrm{Cl}_F(G)$, we can write

$$\mu^{\#} = \sum_{\chi \in \mathrm{Irr}_F(G)} a_\chi \cdot \chi$$

with $a_\chi \in F \ \forall\ \chi \in \mathrm{Irr}_F(G)$. Thus

$$\mu = \mu^{\#}|_{G_{p'}} = \sum_{\chi \in \mathrm{Irr}_F(G)} a_\chi \cdot \chi|_{G_{p'}}$$

where by Lemma 36.1 each $\chi|_{G_{p'}}$ is a Brauer character of G, and hence by Proposition 35.10, each $\chi|_{G_{p'}}$ is a non-negative integer linear combination of irreducible Brauer characters. Therefore μ is an F-linear combination of irreducible Brauer characters over F. It follows that $\mathrm{IBr}_p(G)$ is a generating set for $\mathrm{Cl}_F(G_{p'})$. This proves (a). Parts (b) and (c) are immediate consequences of (a) and its proof. ■

Exercise 36.3

Prove that two kG-modules afford the same Brauer character if and only if they have isomorphic composition factors (including multiplicities).

Remark 36.4

If we translate part (b) of Proposition 36.2 from irreducible characters and Brauer characters to FG-modules and kG-modules, we obtain that the integers $d_{\chi\varphi}$ are the same as the p-decomposition numbers of G as defined through Brauer's reciprocity theorem, i.e.

$$D := \mathrm{Dec}_p(G) = (d_{\chi\varphi})_{\substack{\chi \in \mathrm{Irr}_F(G) \\ \varphi \in \mathrm{IBr}_p(G)}}.$$

The Cartan matrix of G is then

$$C = D^{tr} D = (c_{\varphi\mu})_{\varphi,\mu \in \mathrm{IBr}_p(G)}$$

(see Exercise 34.4) and for every $\varphi, \mu \in \mathrm{IBr}_p(G)$ we have

$$c_{\varphi\mu} = \sum_{\chi \in \mathrm{Irr}_F(G)} d_{\chi\varphi} d_{\chi\mu}.$$

Corollary 36.5

(a) The decomposition matrix $\mathrm{Dec}_p(G)$ of G has full rank, namely $|\mathrm{IBr}_p(G)|$.

(b) The Cartan matrix of G is a symmetric positive definite matrix with non-negative integer entries.

Proof: It follows from Proposition 36.2 that $\{\chi|_{G_{p'}} \mid \chi \in \mathrm{Irr}_F(G)\}$ spans $\mathrm{Cl}_F(G_{p'})$ over F. There is therefore a subset $B \subseteq \{\chi|_{G_{p'}} \mid \chi \in \mathrm{Irr}_F(G)\}$ which forms an F-basis for $\mathrm{Cl}_F(G_{p'})$. Now by Proposition 36.2, the columns of the matrix $(d_{\chi\varphi})_{\chi \in B, \varphi \in \mathrm{IBr}_p(G)}$ are F-linearly independent, hence $\mathrm{Dec}_p(G)$ has full rank. This proves (a) and (b) follows immediately. ■

Recall now that projective kG-modules are liftable by Corollary 32.5. This enables us to associate an F-character of G to each PIM of kG.

Definition 36.6

Let $\varphi \in \mathrm{IBr}_p(G)$ be an irreducible Brauer character afforded by a simple kG-module S. Let P_S be the projective cover of S and let $\widehat{P}_S$ denote a lift of P_S to $\mathcal{O}$. Then, the F-character of $(\widehat{P}_S)^F$ is denoted by Φ_φ and is called the **projective indecomposable character** associated to S or φ.

Corollary 36.7

Let $\varphi \in \mathrm{IBr}_p(G)$. Then:

(a) $\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}_F(G)} d_{\chi\varphi}\chi$; and

(b) $\Phi_\varphi|_{G_{p'}} = \sum_{\mu \in \mathrm{IBr}_p(G)} c_{\varphi\mu}\mu$.

Proof: (a) This follows from Brauer reciprocity.

(b) This follows from part (a) because

$$\Phi_\varphi|_{G_{p'}} = \sum_{\chi \in \mathrm{Irr}_F(G)} d_{\chi\varphi}\chi|_{G_{p'}} = \sum_{\chi \in \mathrm{Irr}_F(G)} d_{\chi\varphi} \sum_{\mu \in \mathrm{IBr}_p(G)} d_{\chi\mu}\mu = \sum_{\mu \in \mathrm{IBr}_p(G)} c_{\varphi\mu}\mu\,.$$

■

Theorem 36.8

Assume $p \nmid |G|$, then the following assertions hold:

(a) If V is a simple FG-module and L is an $\mathcal{O}$-form for V, then its reduction modulo $\mathfrak{p}$ is a simple kG-module and the map

$$\begin{array}{rcl} \mathrm{Irr}_F(G) & \longrightarrow & \mathrm{IBr}_p(G) \\ \chi_V & \mapsto & \chi_V|_{G_{p'}} = \chi_V \end{array}$$

is a bijection.

(b) Both the Cartan matrix of G and the decomposition matrix $\mathrm{Dec}_p(G)$ of G are the identity matrix, provided the rows and the columns are ordered in the same way.

Proof: Since $p \nmid |G|$, clearly $G_{p'} = G$. Now, by Maschke's theorem, kG is semisimple, so the simple kG-modules are precisely the PIMs of kG, up to isomorphism. Thus, if S is a simple kG-module affording the Brauer character φ, then by definition of the Cartan integers (Def. 24.2) and Remark 36.4 we have

$$c_{\varphi\mu} = \delta_{\varphi\mu} \quad \forall\, \mu \in \mathrm{IBr}_p\, G\,.$$

Therefore,

$$1 = c_{\varphi\varphi} = \sum_{\chi \in \mathrm{Irr}_F(G)} d_{\chi\varphi}^2$$

and it follows that there is a unique $\chi_0 \in \mathrm{Irr}_F(G)$ such that $d_{\chi_0\varphi} \neq 0$; in fact we must have $d_{\chi_0\varphi} = 1$. Now, it follows from Corollary 36.7 that

$$\Phi_\varphi = \chi_0 \quad \text{and} \quad \chi_0|_{G_{p'}} = \varphi\,,$$

so the first claim of (a) holds and the given map is well-defined and surjective. It follows immediately that this map is bijective since by Proposition 36.2(c) we have

$$|\operatorname{IBr}_p(G)| = \#G\text{-conjugacy classes in } G = |\operatorname{Irr}_F(G)| < \infty\,.$$

This proves (a) and (b). ■

Finally, we would like to obtain orthogonality relations for Brauer characters, which generalise the row orthogonality relations for ordinary irreducible characters. However, unlike the case of ordinary characters, there are now two significant tables that we can construct: the table of values of Brauer characters of simple modules, and the table of values of Brauer characters of indecomposable projective modules.

Definition 36.9 (*Brauer character table*)

Set $l := |\operatorname{IBr}_p(G)|$ and let $g_1, \dots, g_l$ be a complete set of representatives of the p-regular conjugacy classes of G.

(a) The **Brauer character table** of the finite group G is the matrix $\Big(\varphi(g_j)\Big)_{\substack{\varphi\in\operatorname{IBr}_p(G)\\ 1\leqslant j\leqslant l}} \in M_l(F)\,.$

(b) The **Brauer projective table** of the finite group G at p is the matrix $\Big(\Phi_\varphi(g_j)\Big)_{\substack{\varphi\in\operatorname{IBr}_p(G)\\ 1\leqslant j\leqslant l}} \in M_l(F)\,.$

Remark 36.10

The binary operation

$$\begin{array}{rccl} \langle\, ,\, \rangle_{p'}: & \operatorname{Cl}_F(G_{p'}) \times \operatorname{Cl}_F(G_{p'}) & \longrightarrow & F \\ & (f_1, f_2) & \mapsto & \langle f_1, f_2\rangle_{p'} := \frac{1}{|G|}\sum_{g\in G_{p'}} f_1(g) f_2(g^{-1}) \end{array}$$

is a Hermitian form on $\operatorname{Cl}_F(G_{p'})$.

With these tools we obtain a replacement for the row orthogonality relations for ordinary irreducible characters. It says that the rows of the Brauer character table and of the p-projective Brauer table are orthogonal to each other.

Theorem 36.11

(a) All projective indecomposable characters Φ_φ ($\varphi \in \operatorname{IBr}_p(G)$) vanish on p-singular elements, and the set $\{\Phi_\varphi \mid \varphi\in\operatorname{IBr}_p(G)\}$ is an F-basis of the subspace

$$\operatorname{Cl}^\circ_F(G) := \{f \in \operatorname{Cl}_F(G) \mid f(g) = 0\ \forall\, g\in G\setminus G_{p'}\}$$

of $\operatorname{Cl}_F(G)$.

(b) For every $\varphi, \psi \in \operatorname{IBr}_p(G)$ we have $\langle \varphi, \Phi_\psi\rangle_{p'} = \delta_{\varphi\psi} = \langle \Phi_\varphi, \psi\rangle_{p'}\,.$

Proof: Let $g_1, \dots, g_r$ be a complete set of representatives of the conjugacy classes of G such that $g_1, \dots, g_l$ are p-singular and $g_{l+1}, \dots, g_r$ are p-regular. If $l < j \leqslant r$, then by Proposition 36.2(b),

$$\chi(g_j^{-1}) = \chi|_{G_{p'}}(g_j^{-1}) = \sum_{\varphi\in\operatorname{IBr}_p(G)} d_{\chi\varphi}\varphi(g_j^{-1})$$

and for each $1 \leqslant i \leqslant r$ the 2nd Orthogonality Relations for the irreducible F-characters (see [Las20, Thm. 12.2]) yield

$$(*) \qquad \delta_{ij}|C_G(g_i)| = \sum_{\chi \in \mathrm{Irr}_F(G)} \chi(g_i)\chi(g_j^{-1}) = \sum_{\chi \in \mathrm{Irr}_F(G)} \chi(g_i)\Big(\sum_{\varphi \in \mathrm{IBr}_p(G)} d_{\chi\varphi}\varphi(g_j^{-1})\Big)$$
$$= \sum_{\varphi \in \mathrm{IBr}_p(G)} \Big(\sum_{\chi \in \mathrm{Irr}_F(G)} d_{\chi\varphi}\chi(g_i)\Big)\varphi(g_j^{-1})$$
$$= \sum_{\varphi \in \mathrm{IBr}_p(G)} \Phi_\varphi(g_i)\varphi(g_j^{-1})$$

where the last equality holds by Corollary 36.7(a). Then for each $1 \leqslant i \leqslant l$, we have $\sum_{\varphi \in \mathrm{IBr}_p(G)} \Phi_\varphi(g_i)\varphi = 0$ in $\mathrm{Cl}_F(G_{p'})$ and we obtain that $\Phi_\varphi(g_i) = 0$ for each $\varphi \in \mathrm{IBr}_p(G)$ since $\mathrm{IBr}_p(G)$ is F-linearly independent. Thus the first claim of (a) is established.
Now, $(*)$ says that the square matrix

$$\left(\frac{\Phi_\varphi(g_i)}{|C_G(g_i)|}\right)^{tr}_{\substack{\varphi \in \mathrm{IBr}_p(G) \\ l+1 \leqslant i \leqslant r}}$$

is a left inverse for the square matrix

$$\Big(\varphi(g_i^{-1})\Big)_{\substack{\varphi \in \mathrm{IBr}_p(G) \\ l+1 \leqslant i \leqslant r}}.$$

Therefore, it is also a right inverse, and multiplying both matrices the other way around, we obtain (applying the Orbit-Stabiliser Theorem) that

$$\delta_{\varphi\psi} = \sum_{i=l+1}^{r} \frac{1}{|C_G(g_i)|}\varphi(g_i^{-1})\Phi_\psi(g_i) = \frac{1}{|G|}\sum_{g \in G_{p'}} \varphi(g^{-1})\Phi_\psi(g)$$
$$= \frac{1}{|G|}\sum_{g \in G_{p'}} \varphi(g)\Phi_\psi(g^{-1})$$
$$= \langle \varphi, \Phi_\psi \rangle_{p'}$$

for all $\varphi, \psi \in \mathrm{IBr}_p(G)$. Similarly $\langle \Phi_\varphi, \psi \rangle_{p'} = \delta_{\varphi\psi}$ for all $\varphi, \psi \in \mathrm{IBr}_p(G)$. This means in effect that $\{\Phi_\varphi|_{G_{p'}} \mid \varphi \in \mathrm{IBr}_p(G)\}$ and $\mathrm{IBr}_p(G)$ are dual bases of $\mathrm{Cl}_F(G_{p'})$ with respect to the Hermitian form $\langle , \rangle_{p'}$. Thus, we have proved (a) and (b). ∎

Remark 36.12

The proof of the theorem tells us that writing Φ for the Brauer projective table, Π for the Brauer character table and setting $B := \mathrm{diag}(|C_G(g_{l+1})|, \ldots, |C_G(g_r)|)$, then the orthogonality relations can be written as $\Phi^{tr}B^{-1}\overline{\Pi} = I$.

Exercise 36.13

Let H be a p'-subgroup of a finite group G. Prove that the character Φ_k is a constituent of the trivial F-character of H induced to G.

Exercise 36.14

Let $\varphi \in \mathrm{IBr}_p(G)$ and let λ be a linear character. Prove that $\lambda\varphi \in \mathrm{IBr}_p(G)$ and $\lambda(\Phi_\varphi)|_{G_{p'}} = (\Phi_{\lambda\varphi})|_{G_{p'}}$.

Exercise 36.15

Let G be a finite group and let ρ_{reg} denote the regular F-character of G. Prove that:

$$\rho_{\mathrm{reg}} = \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)\Phi_\varphi \quad \text{and} \quad (\rho_{\mathrm{reg}})|_{G_{p'}} = \sum_{\varphi \in \mathrm{IBr}_p(G)} \Phi_\varphi(1)\varphi\,.$$

Exercise 36.16

Deduce from Theorem 36.11 that:

(a) the inverse of the Cartan matrix of kG is $C^{-1} = (\langle \varphi, \psi \rangle_{p'})_{\varphi,\psi \in \mathrm{IBr}_p(G)}$; and

(b) $|G|_p \mid \Phi_\varphi(1)$ for every $\varphi \in \mathrm{IBr}_p(G)$.

Exercise 36.17

(a) Let U be a kG-module and let P be a PIM of kG. Prove that

$$\dim_k \mathrm{Hom}_{kG}(P, U) = \frac{1}{|G|} \sum_{g \in G_{p'}} \varphi_P(g^{-1})\varphi_U(g)$$

(b) Prove that the binary operation $\langle\, , \rangle_{p'}$ is an inner product on $\mathrm{Cl}_F(G_{p'})$.

Exercise 36.18

Let $G := \mathfrak{A}_5$, the alternating group on 5 letters. Calculate the Brauer character table, the Cartan matrix and the decomposition matrix of G for $p = 3$.

[Hints. (1.) Use the ordinary character table of $\mathfrak{A}_5$ and reduction modulo p. (2.) A simple group does not have any irreducible Brauer character of degree 2.]

Exercise 36.19

Deduce from Remark 36.12 that column orthogonality relations for the Brauer characters take the form $\overline{\sqcap}^{tr}\Phi = B$, i.e. given $g, h \in G_{p'}$ we have

$$\sum_{\phi \in \mathrm{IBr}_p(G)} \phi(g)\Phi_\varphi(h^{-1}) = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are } G\text{-conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 36.20

Let $\sigma : G \longrightarrow H$ be an isomorphism of groups. If ψ is a class function defined on G (resp. on $G_{p'}$), we define

$$\psi^\sigma(x) := \psi(x^{\sigma^{-1}}) \qquad \forall\, x \in H \text{ (resp. } \forall\, x \in H_{p'})$$

Prove that ψ is a Brauer character of G if and only if ψ^σ is a Brauer character of H. Prove that moreover $d_{\chi\varphi} = d_{\chi^\sigma\varphi^\sigma}$ for every $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(G)$.

Exercise 36.21

Let $N \trianglelefteq G$ be the smallest normal subgroup of G such that G/N is an abelian p'-group. Prove that the map

$$\begin{array}{rccc} \Psi: & \mathrm{Irr}(G/N) & \longrightarrow & \{\lambda \in \mathrm{IBr}_p(G) \mid \lambda(1) = 1\} \\ & \chi & \mapsto & \mathrm{Inf}_{G/N}^{G}(\chi)|_{G_{p'}} \end{array}$$

is a bijection and conclude that the number of linear Brauer characters of G is $(G/[G:G])_{p'}$.