Chapter 1. Foundations of Representation Theory

In this chapter we review four important module-theoretic theorems, which lie at the foundations of *representation theory of finite groups*:

1. **Schur's Lemma**: about homomorphisms between simple modules.
2. **The Jordan-Hölder Theorem**: about "uniqueness" properties of composition series.
3. **Nakayama's Lemma**: about an essential property of the Jacobson radical.
4. **The Krull-Schmidt Theorem**: about direct sum decompositions into indecomposable submodules.

Notation: throughout this chapter, unless otherwise specified, we let R denote an arbitrary unital and associative ring.

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1 (Ir)Reducibility and (in)decomposability

Submodules and direct sums of modules allow us to introduce the two main notions that will enable us to break modules in *elementary* pieces in order to simplify their study: *simplicity* and *indecomposability*.

Definition 1.1 (*simple/irreducible module / indecomposable module / semisimple module*)

(a) An R-module M is called **reducible** if it admits an R-submodule U such that $0 \subsetneq U \subsetneq M$. An R-module M is called **simple**, or **irreducible**, if it is non-zero and not reducible. We let $\operatorname{Irr}(R)$ denote a set of representative of the isomorphism classes of simple R-modules.

(b) An R-module M is called **decomposable** if M possesses two non-zero proper submodules M_1, M_2 such that $M = M_1 \oplus M_2$. An R-module M is called **indecomposable** if it is non-zero and not decomposable.

(c) An R-module M is called **completely reducible** or **semisimple** if it admits a direct sum decomposition into simple R-submodules.

Our primary goal in Chapter 1 and Chapter 2 is to investigate each of these three concepts in details.

Remark 1.2

Clearly any simple module is also indecomposable, resp. semisimple. However, the converse does not hold in general.

Exercise 1.3

Prove that if $(R, +, \cdot)$ is a ring, then $R^\circ := R$ itself maybe seen as an R-module via left multiplication in R, i.e. where the external composition law is given by

$$R \times R^\circ \longrightarrow R^\circ, (r, m) \mapsto r \cdot m\,.$$

We call R° the **regular** R-module. Prove that:

(a) the R-submodules of R° are precisely the left ideals of R;

(b) $I \lhd R$ is a maximal left ideal of $R \Leftrightarrow R^\circ/I$ is a simple R-module, and $I \lhd R$ is a minimal left ideal of $R \Leftrightarrow I$ is simple when regarded as an R-submodule of R°.

2 Schur's Lemma

Schur's Lemma is a basic result, which lets us understand homomorphisms between *simple* modules, and, more importantly, endomorphisms of such modules.

Theorem 2.1 (*Schur's Lemma*)

(a) Let V, W be simple R-modules. Then:

  (i) $\operatorname{End}_R(V)$ is a skew-field, and

  (ii) if $V \not\cong W$, then $\operatorname{Hom}_R(V, W) = 0$.

(b) If K is an algebraically closed field, A is a K-algebra, and V is a simple A-module such that $\dim_K V < \infty$, then

$$\operatorname{End}_A(V) = \{\lambda \operatorname{Id}_V \mid \lambda \in K\} \cong K\,.$$

Proof:

(a) First, we claim that every $f \in \mathrm{Hom}_R(V, W)\setminus\{0\}$ admits an inverse in $\mathrm{Hom}_R(W, V)$.
Indeed, $f \neq 0 \implies \ker f \subsetneq V$ is a proper R-submodule of V and $\{0\} \neq \mathrm{Im} f$ is a non-zero R-submodule of W. But then, on the one hand, $\ker f = \{0\}$, because V is simple, hence f is injective, and on the other hand, $\mathrm{Im} f = W$ because W is simple. It follows that f is also surjective, hence bijective. Therefore, by Example ??(d), f is invertible with inverse $f^{-1} \in \mathrm{Hom}_R(W, V)$.
Now, (ii) is straightforward from the above. For (i), first recall that $\mathrm{End}_R(V)$ is a ring, which is obviously non-zero as $\mathrm{End}_R(V) \ni \mathrm{Id}_V$ and $\mathrm{Id}_V \neq 0$ because $V \neq 0$ since it is simple. Thus, as any $f \in \mathrm{End}_R(V)\setminus\{0\}$ is invertible, $\mathrm{End}_R(V)$ is a skew-field.

(b) Let $f \in \mathrm{End}_A(V)$. By the assumptions on K, f has an eigenvalue $\lambda \in K$. Let $v \in V\setminus\{0\}$ be an eigenvector of f for λ. Then $(f - \lambda\,\mathrm{Id}_V)(v) = 0$. Therefore, $f - \lambda\,\mathrm{Id}_V$ is not invertible and

$$f - \lambda\,\mathrm{Id}_V \in \mathrm{End}_A(V) \quad \overset{(a)}{\Longrightarrow} \quad f - \lambda\,\mathrm{Id}_V = 0 \quad \Longrightarrow \quad f = \lambda\,\mathrm{Id}_V\,.$$

Hence $\mathrm{End}_A(V) \subseteq \{\lambda\,\mathrm{Id}_V \mid \lambda \in K\}$, but the reverse inclusion also obviously holds, so that

$$\mathrm{End}_A(V) = \{\lambda\,\mathrm{Id}_V\} \cong K\,.$$

■

3 Composition series and the Jordan-Hölder Theorem

From Chapter 2 on, we will assume that all modules we work with can be broken into *simple* modules in the sense of the following definition.

Definition 3.1 (*Composition series / composition factors / composition length*)

Let M be an R-module.

(a) A **series** (or **filtration**) of M is a finite chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \qquad (n \in \mathbb{Z}_{\geqslant 0})\,.$$

(b) A **composition series** of M is a series

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \qquad (n \in \mathbb{Z}_{\geqslant 0})$$

where M_i/M_{i-1} is simple for each $1 \leqslant i \leqslant n$. The quotient modules M_i/M_{i-1} are called the **composition factors** (or the **constituents**) of M and the integer n is called the **composition length** of M.

Notice that, clearly, in a composition series all inclusions are in fact strict because the quotient modules are required to be simple, hence non-zero.

Next we see that the existence of a *composition series* implies that the module is *finitely generated*. However, the converse does not hold in general. This is explained through the fact that the existence of a composition series is equivalent to the fact that the module is both *Noetherian* and *Artinian*.

Definition 3.2 (*Chain conditions / Artinian and Noetherian rings and modules*)

(a) An R-module M is said to satisfy the **descending chain condition** (D.C.C.) on submodules (or to be **Artinian**) if every descending chain $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_r \supseteq \ldots \supseteq \{0\}$ of submodules eventually becomes stationary, i.e. $\exists\ m_0$ such that $M_m = M_{m_0}$ for every $m \geqslant m_0$.

(b) An R-module M is said to satisfy the **ascending chain condition** (A.C.C.) on submodules (or to be **Noetherian**) if every ascending chain $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r \subseteq \ldots \subseteq M$ of submodules eventually becomes stationary, i.e. $\exists\ m_0$ such that $M_m = M_{m_0}$ for every $m \geqslant m_0$.

(c) The ring R is called **left Artinian** (resp. **left Noetherian**) if the regular module R° is Artinian (resp. Noetherian).

Theorem 3.3 (*Jordan-Hölder*)

Any series of R-submodules $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ ($r \in \mathbb{Z}_{\geqslant 0}$) of an R-module M may be refined to a composition series of M. In addition, if

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M \quad (n \in \mathbb{Z}_{\geqslant 0})$$

and

$$0 = M'_0 \subsetneq M'_1 \subsetneq \ldots \subsetneq M'_m = M \quad (m \in \mathbb{Z}_{\geqslant 0})$$

are two composition series of M, then $m = n$ and there exists a permutation $\pi \in \mathfrak{S}_n$ such that $M'_i/M'_{i-1} \cong M_{\pi(i)}/M_{\pi(i)-1}$ for every $1 \leqslant i \leqslant n$. In particular, the composition length is well-defined.

Proof: See *Commutative Algebra*. ■

Corollary 3.4

If M is an R-module, then TFAE:

(a) M has a composition series;

(b) M satisfies D.C.C. and A.C.C. on submodules;

(c) M satisfies D.C.C. on submodules and every submodule of M is finitely generated.

Proof: See *Commutative Algebra*. ■

Theorem 3.5 (*Hopkins' Theorem*)

If M is a module over a left Artinian ring, then TFAE:

(a) M has a composition series;

(b) M satisfies D.C.C. on submodules;

(c) M satisfies A.C.C. on submodules;

(d) M is finitely generated.

Proof: Without proof. ■

4 The Jacobson radical and Nakayama's Lemma

The Jacobson radical is one of the most important two-sided ideals of a ring. As we will see in the next sections and Chapter 2, this ideal carries a lot of information about the structure of a ring and that of its modules.

Proposition-Definition 4.1 (*Annihilator / Jacobson radical*)

(a) Let M be an R-module. Then $\operatorname{ann}_R(M) := \{r \in R \mid rm = 0 \ \ \forall\, m \in M\}$ is a two-sided ideal of R, called the **annihilator** of M.

(b) The **Jacobson radical** of R is the two-sided ideal

$$J(R) := \bigcap_{V \in \operatorname{Irr}(R)} \operatorname{ann}_R(V) = \{x \in R \mid 1 - axb \in R^\times \ \ \forall\, a, b \in R\}\,.$$

(c) If V is a simple R-module, then there exists a maximal left ideal $I \lhd R$ such that $V \cong R^\circ/I$ (as R-modules) and

$$J(R) = \bigcap_{\substack{I \lhd R,\\ I \text{ maximal}\\ \text{left ideal}}} I\,.$$

Proof: See *Commutative Algebra.* ■

Exercise 4.2

(a) Prove that any simple R-module may be seen as a simple $R/J(R)$-module.

(b) Conversely, prove that any simple $R/J(R)$-module may be seen as a simple R-module. [Hint: use a change of the base ring via the canonical morphism $R \longrightarrow R/J(R)$.]

(c) Deduce that R and $R/J(R)$ have the same simple modules.

Theorem 4.3 (*Nakayama's Lemma*)

If M is a finitely generated R-module and $J(R)M = M$, then $M = 0$.

Proof: See *Commutative Algebra.* ■

Remark 4.4

One often needs to apply Nakayama's Lemma to a finitely generated quotient module M/U, where U is an R-submodule of M. In that case the result may be restated as follows:

$$M = U + J(R)M \quad \Longrightarrow \quad U = M$$

5 Indecomposability and the Krull-Schmidt Theorem

We now consider the notion of *indecomposability* in more details. Our first aim is to prove that indecomposability can be recognised at the endomorphism algebra of a module.

Definition 5.1

A ring R is said to be **local** $:\Longleftrightarrow R\backslash R^\times$ is a two-sided ideal of R.

Example 1

(a) Any field K is local because $K\backslash K^\times = \{0\}$ by definition.

(b) Exercise: Let p be a prime number and $R := \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$. Prove that $R\backslash R^\times = \{\frac{a}{b} \in R \mid p|a\}$ and deduce that R is local.

(c) Exercise: Let K be a field and let $R := \left\{A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \dots & a_{n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{pmatrix} \in M_n(K)\right\}$. Prove that $R\backslash R^\times = \{A \in R \mid a_1 = 0\}$ and deduce that R is local.

Proposition 5.2

Let R be a ring. Then TFAE:

(a) R is local;

(b) $R\backslash R^\times = J(R)$, i.e. $J(R)$ is the unique maximal left ideal of R;

(c) $R/J(R)$ is a skew-field.

Proof: Set $N := R\backslash R^\times$.

(a)⇒(b): Clear: $I \lhd R$ proper left ideal $\Rightarrow I \subseteq N$. Hence, by Proposition-Definition 4.1(c),

$$J(R) = \bigcap_{\substack{I \lhd R,\\ I \text{ maximal} \\ \text{left ideal}}} I \subseteq N\,.$$

Now, by (a) N is an ideal of R, hence N must be a maximal left ideal, even the unique one. It follows that $N = J(R)$.

(b)⇒(c): If $J(R)$ is the unique maximal left ideal of R, then in particular $R \neq 0$ and $R/J(R) \neq 0$. So let $r \in R\backslash J(R) \overset{(b)}{=} R^\times$. Then obviously $r + J(R) \in (R/J(R))^\times$. It follows that $R/J(R)$ is a skew-field.

(c)⇒(a): Since $R/J(R)$ is a skew-field by (c), $R/J(R) \neq 0$, so that $R \neq 0$ and there exists $a \in R\backslash J(R)$. Moreover, again by (c), $a + J(R) \in (R/J(R))^\times$, so that $\exists\, b \in R\backslash J(R)$ such that

$$ab + J(R) = 1 + J(R) \in R/J(R)$$

Therefore, $\exists\, c \in J(R)$ such that $ab = 1 - c$, which is invertible in R by Proposition-Definition 4.1(b). Hence $\exists\, d \in R$ such that $abd = (1-c)d = 1 \Rightarrow a \in R^\times$. Therefore $R\backslash J(R) = R^\times$, and it follows that $R\backslash R^\times = J(R)$ which is a two-sided ideal of R. ■

Proposition 5.3 (*Fitting's Lemma*)

Let M be an R-module which has a composition series and let $\varphi \in \mathrm{End}_R(M)$ be an endomorphism of M. Then there exists $n \in \mathbb{Z}_{>0}$ such that

(i) $\varphi^n(M) = \varphi^{n+i}(M)$ for every $i \geqslant 1$;

(ii) $\ker(\varphi^n) = \ker(\varphi^{n+i})$ for every $i \geqslant 1$; and

(iii) $M = \varphi^n(M) \oplus \ker(\varphi^n)$.

Proof: By Corollary 3.4 the module M satisfies both A.C.C. and D.C.C. on submodules. Hence the two chains of submodules

$$\varphi(M) \supseteq \varphi^2(M) \supseteq \dots,$$
$$\ker(\varphi) \subseteq \ker(\varphi^2) \subseteq \dots$$

eventually become stationary. Therefore we can find an index n satisfying both (i) and (ii).

Exercise: Prove that $M = \varphi^n(M) \oplus \ker(\varphi^n)$. ■

Proposition 5.4

Let M be an R-module which has a composition series. Then:

$$M \text{ is indecomposable} \iff \mathrm{End}_R(M) \text{ is a local ring.}$$

Proof: "⇒": Assume that M is indecomposable. Let $\varphi \in \mathrm{End}_R(M)$. Then by Fitting's Lemma there exists $n \in \mathbb{Z}_{>0}$ such that $M = \varphi^n(M) \oplus \ker(\varphi^n)$. As M is indecomposable either $\varphi^n(M) = M$ and $\ker(\varphi^n) = 0$ or $\varphi^n(M) = 0$ and $\ker(\varphi^n) = M$.

- In the first case φ is bijective, hence invertible.
- In the second case φ is nilpotent.

Therefore, $N := \mathrm{End}_R(M) \setminus \mathrm{End}_R(M)^\times = \{\text{nilpotent elements of } \mathrm{End}_R(M)\}$.

Claim: N is a two-sided ideal of $\mathrm{End}_R(M)$.

Let $\varphi \in N$ and $m \in \mathbb{Z}_{>0}$ minimal such that $\varphi^m = 0$. Then

$$\varphi^{m-1}(\varphi\rho) = 0 = (\rho\varphi)\varphi^{m-1} \quad \forall\, \rho \in \mathrm{End}_R(M).$$

As $\varphi^{m-1} \neq 0$, $\varphi\rho$ and $\rho\varphi$ cannot be invertible, hence $\varphi\rho, \rho\varphi \in N$.

Next let $\varphi, \rho \in N$. If $\varphi + \rho =: \psi$ were invertible in $\mathrm{End}_R(M)$, then by the previous argument we would have $\psi^{-1}\rho, \psi^{-1}\varphi \in N$, which would be nilpotent. Hence

$$\psi^{-1}\varphi = \psi^{-1}(\psi - \rho) = \mathrm{Id}_M - \psi^{-1}\rho$$

would be invertible.
(Indeed, $\psi^{-1}\rho$ nilpotent $\Rightarrow (\mathrm{Id}_M - \psi^{-1}\rho)(\mathrm{Id}_M + \psi^{-1}\rho + (\psi^{-1}\rho)^2 + \dots + (\psi^{-1}\rho)^{a-1}) = \mathrm{Id}_M$, where a is minimal such that $(\psi^{-1}\rho)^a = 0$.)
This is a contradiction. Therefore $\varphi + \rho \in N$, which proves that N is an ideal.

Finally, it follows from the Claim and the definition that $\mathrm{End}_R(M)$ is local.

"⇐": Assume M is decomposable and let M_1, M_2 be proper submodules such that $M = M_1 \oplus M_2$. Then consider the two projections

$$\pi_1 : M_1 \oplus M_2 \longrightarrow M_1 \oplus M_2, (m_1, m_2) \mapsto (m_1, 0)$$

onto M_1 along M_2 and

$$\pi_2 : M_1 \oplus M_2 \longrightarrow M_1 \oplus M_2, (m_1, m_2) \mapsto (0, m_2)$$

onto M_2 along M_1. Clearly $\pi_1, \pi_2 \in \mathrm{End}_R(M)$ but $\pi_1, \pi_2 \notin \mathrm{End}_R(M)^\times$ since they are not surjective by construction. Now, as $\pi_2 = \mathrm{Id}_M - \pi_1$ is not invertible it follows from the characterisation of the Jacobson radical of Proposition-Definition 4.1(b) that $\pi_1 \notin J(\mathrm{End}_R(M))$. Therefore

$$\mathrm{End}_R(M) \setminus \mathrm{End}_R(M)^\times \neq J(\mathrm{End}_R(M))$$

and it follows from Proposition 5.2 that $\mathrm{End}_R(M)$ is not a local ring. ■

Next, we want to be able to decompose R-modules into direct sums of indecomposable submodules. The Krull-Schmidt Theorem will then provide us with certain uniqueness properties of such decompositions.

Proposition 5.5

Let M be an R-module. If M satisfies either A.C.C. or D.C.C., then M admits a decomposition into a direct sum of finitely many indecomposable R-submodules.

Proof: Let us assume that M is not expressible as a finite direct sum of indecomposable submodules. Then in particular M is decomposable, so that we may write $M = M_1 \oplus W_1$ as a direct sum of two proper submodules. W.l.o.g. we may assume that the statement is also false for W_1. Then we also have a decomposition $W_1 = M_2 \oplus W_2$, where M_2 and W_2 are proper sumbodules of W_1 with the statement being false for W_2. Iterating this argument yields the following infinite chains of submodules:

$$W_1 \supsetneq W_2 \supsetneq W_3 \supsetneq \cdots,$$

$$M_1 \subsetneq M_1 \oplus M_2 \subsetneq M_1 \oplus M_2 \oplus M_3 \subsetneq \cdots.$$

The first chain contradicts D.C.C. and the second chain contradicts A.C.C.. The claim follows. ■

Theorem 5.6 (*Krull–Schmidt*)

Let M be an R-module which has a composition series. If

$$M = M_1 \oplus \cdots \oplus M_n = M'_1 \oplus \cdots \oplus M'_{n'} \qquad (n, n' \in \mathbb{Z}_{>0})$$

are two decomposition of M into direct sums of finitely many indecomposable R-submodules, then $n = n'$, and there exists a permutation $\pi \in \mathfrak{S}_n$ such that $M_i \cong M'_{\pi(i)}$ for each $1 \leqslant i \leqslant n$ and

$$M = M'_{\pi(1)} \oplus \cdots \oplus M'_{\pi(r)} \oplus \bigoplus_{j=r+1}^{n} M_j \qquad \text{for every } 1 \leqslant r \leqslant n.$$

Proof: For each $1 \leqslant i \leqslant n$ let

$$\pi_i : M = M_1 \oplus \cdots \oplus M_n \to M_i, m_1 + \ldots + m_n \mapsto m_i$$

be the projection on the i-th factor of first decomposition, and for each $1 \leqslant j \leqslant n'$ let

$$\psi_j : M = M'_1 \oplus \cdots \oplus M'_{n'} \to M'_j, m'_1 + \ldots + m'_{n'} \mapsto m'_j$$

be the projection on the j-th factor of second decomposition.

Claim: if $\psi \in \mathrm{End}_R(M)$ is such that $\pi_1 \circ \psi|_{M_1} : M_1 \to M_1$ is an isomorphism, then

$$M = \psi(M_1) \oplus M_2 \oplus \cdots \oplus M_n \text{ and } \psi(M_1) \cong M_1 .$$

Indeed: By the assumption of the claim, both $\psi|_{M_1} : M_1 \to \psi(M_1)$ and $\pi_1|_{\psi(M_1)} : \psi(M_1) \to M_1$ must be isomorphisms. Therefore $\psi(M_1) \cap \ker(\pi_1) = 0$, and for every $m \in M$ there exists $m'_1 \in \psi(M_1)$ such that $\pi_1(m) = \pi_1(m'_1)$, hence $m - m'_1 \in \ker(\pi_1)$. It follows that

$$M = \psi(M_1) + \ker(\pi_1) = \psi(M_1) \oplus \ker(\pi_1) = \psi(M_1) \oplus M_2 \oplus \cdots \oplus M_n .$$

Hence the Claim holds.

Now, we have $\mathrm{Id}_M = \sum_{j=1}^{n'} \psi_j$, and so $\mathrm{Id}_{M_1} = \sum_{j=1}^{n'} \pi_1 \circ \psi_j|_{M_1} \in \mathrm{End}_R(M_1)$. But as M has a composition series, so has M_1, and therefore $\mathrm{End}_R(M_1)$ is local by Proposition 5.4. Thus if all the $\pi_1 \circ \psi_j|_{M_1} \in \mathrm{End}_R(M_1)$ are not invertible, they are all nilpotent and then so is Id_{M_1}, which is in turn not invertible. This is not possible, hence it follows that there exists an index j such that

$$\pi_1 \circ \psi_j|_{M_1} : M_1 \to M_1$$

is an isomorphism and the Claim implies that $M = \psi_j(M_1) \oplus M_2 \oplus \cdots \oplus M_n$ and $\psi_j(M_1) \cong M_1$.
We then set $\pi(1) := j$. By definition $\psi_j(M_1) \subseteq M'_j$ as M'_j is indecomposbale, so that

$$\psi_j(M_1) \cong M'_j = M'_{\pi(1)} .$$

Finally, an induction argument (Exercise!) yields:

$$M = M'_{\pi(1)} \oplus \cdots \oplus M'_{\pi(r)} \oplus \bigoplus_{j=r+1}^{n} M_j ,$$

mit $M'_{\pi(i)} \cong M_i$ $(1 \leqslant i \leqslant r)$. In particular, the case $r = n$ implies the equality $n = n'$. ■