Chapter 8. Indecomposable Modules

After simple and projective modules, the goal of this chapter is to understand indecomposable modules in general. Apart for exceptions, the group algebra is of *wild representation type*, which, roughly speaking, means that it is not possible to classify the indecomposable modules over such algebras. However, representation theorists have developed tools which enable us to organise indecomposable modules in packages parametrised by parameters that are useful enough to understand essential properties of these modules. In this respect, first we will generalise the idea of a projective module seen in Chapter 7 by defining what is called **relative projectivity**. This will lead us to introduce the concepts of **vertices** and **sources** of indecomposable modules, which are two typical examples of parameters bringing us useful information about indecomposable modules in general.

Notation: throughout this chapter, unless otherwise specified, we let G denote a finite group. We let $(F, \mathcal{O}, k)$ be a p-modular system and $K \in \{F, \mathcal{O}, k\}$. All KG-modules considered are assumed to be **free of finite rank as K-modules**.

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27 Relative projectivity

Relative projectivity is a refinement of the idea of projectivity seen in Chapter 7, exploiting induction and restriction from subgroups.

Definition 27.1

Let $H \leqslant G$.

(a) A KG-module M is called H**-free** if there exists a KH-module V such that $M \cong V\uparrow_H^G$.

(b) A KG-module M is called **relatively** H**-projective**, or simply H**-projective**, if it is isomorphic to a direct summand of an H-free module, i.e. if there exists a KH-module V such that $M \mid V\uparrow_H^G$.

Remark 27.2

It is easy to see that H-freeness is a generalisation of freeness and relative projectivity is a generalisation of projectivity.

(1) Freeness is the same as $\{1\}$-freeness: indeed, as $KG \cong K\uparrow_{\{1\}}^G$ by Example 10, clearly $(KG)^n \cong (K^n)\uparrow_{\{1\}}^G$.

(2) Projectivity is the same as $\{1\}$-projectivity: a KG-module is projective $\Leftrightarrow$ it is a direct summand of a free KG-module $\Leftrightarrow$ it is a direct summand of a $\{1\}$-free KG-module $\Leftrightarrow$ it is relatively $\{1\}$-projective.

To begin with, we would like to characterise relative projectivity in a similar way we characterised projectivity in Proposition-Definition B.5. To reach this aim, we first take a closer look at the adjunction between induction and restriction, we have seen in Theorem 17.10.

Notation 27.3

Let $H \leqslant G$.

(1) Let $\varphi : U_1 \longrightarrow U_2$ be a KH-homomorphism. Then we denote by $\varphi\uparrow_H^G$ the induced KG-homomorhpism

$$\begin{aligned}\varphi\uparrow_H^G := \mathrm{Id}_{KG} \otimes \varphi : U_1\uparrow_H^G = KG \otimes_{KH} U_1 &\longrightarrow U_2\uparrow_H^G = KG \otimes_{KH} U_2 \\ x \otimes u &\mapsto x \otimes \varphi(u)\,.\end{aligned}$$

(2) Let U be a KH-module and V be a KG-module. The K-isomorphisms

$$\Phi := \Phi_{U,V} : \mathrm{Hom}_{KG}(U\uparrow_H^G, V) \xrightarrow{\cong} \mathrm{Hom}_{KH}(U, V\downarrow_H^G)$$

and

$$\Psi := \Psi_{U,V} : \mathrm{Hom}_{KH}(U, V\downarrow_H^G) \xrightarrow{\cong} \mathrm{Hom}_{KG}(U\uparrow_H^G, V)$$

from Theorem 17.10 tell us that the induction and restriction functors Ind_H^G and Res_H^G form a pair of bi-adjoint functors. The first isomorphism translates the fact that Ind_H^G is *left adjoint* to Res_H^G and the second isomorphism translates the fact that Ind_H^G is *right adjoint* to Res_H^G.

Explained in more details, there may of course be many such isomorphisms, but there is a choice which is called *natural* in U and V. Spelled out, this means that whenever a morphism $\gamma \in \mathrm{Hom}_{KH}(U_1, U_2)$ is given, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{KG}(U_1\uparrow_H^G, V) \arrow[r, "\Phi_{U_1,V}", "\cong"'] & \mathrm{Hom}_{KH}(U_1, V\downarrow_H^G) \\
\mathrm{Hom}_{KG}(U_2\uparrow_H^G, V) \arrow[r, "\cong", "\Phi_{U_2,V}"'] \arrow[u, "(\gamma\uparrow_H^G)^*"] & \mathrm{Hom}_{KH}(U_2, V\downarrow_H^G) \arrow[u, "\gamma^*"']
\end{tikzcd}$$

commutes and whenever $\alpha \in \mathrm{Hom}_{KG}(V_1, V_2)$ is given, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{KH}(U, V_1\downarrow_H^G) \arrow[r, "\Psi_{U,V_1}", "\cong"'] \arrow[d, "\alpha_*"'] & \mathrm{Hom}_{KG}(U\uparrow_H^G, V_1) \arrow[d, "\alpha_*"] \\
\mathrm{Hom}_{KH}(U, V_2\downarrow_H^G) \arrow[r, "\cong", "\Psi_{U,V_2}"'] & \mathrm{Hom}_{KG}(U\uparrow_H^G, V_2)
\end{tikzcd}$$

commutes. (For the upper and lower $*$ notation, see again Proposition D.3.) For the case Ind_H^G is *right adjoint* to Res_H^G similar diagrams must commute. (Exercise: write down these diagrams!)
In order to understand relative H-projectivity, we consider the **unit** and the **counit** of the adjunction saying that Ind_H^G is *left adjoint* to Res_H^G, i.e. the KH-homomorphism

$$\begin{aligned}\mu : U \longrightarrow U\uparrow_H^G\downarrow_H^G = \bigoplus_{g\in[G/H]} g\otimes U = 1\otimes U \;\oplus \bigoplus_{g\in[G/H], g\neq 1} g\otimes U \\ u \mapsto 1\otimes u\end{aligned}$$

(i.e. the natural inclusion of U into the summand $1\otimes U$) and the KG-homomorphism

$$\begin{aligned}\varepsilon : V\downarrow_H^G\uparrow_H^G = \bigoplus_{g\in[G/H]} g\otimes(V\downarrow_H^G) \longrightarrow V \\ g\otimes v \mapsto gv\,.\end{aligned}$$

For any $u \in U$, we have $\varepsilon\circ\mu(u) = \varepsilon(1\otimes u) = u$, so $\varepsilon\circ\mu = \mathrm{Id}_U$ and thus we deduce that:

- μ is a KH-section for ε;
- μ is injective; and
- ε is surjective.

This yields the mutually inverse natural K-isomorphisms

$$\Phi = \Phi_{U,V} : \mathrm{Hom}_{KG}(U\uparrow_H^G, V) \longrightarrow \mathrm{Hom}_{KH}(U, V\downarrow_H^G), \psi \mapsto \psi\circ\mu\,,$$

$$\Psi = \Psi_{U,V} : \mathrm{Hom}_{KH}(U, V\downarrow_H^G) \longrightarrow \mathrm{Hom}_{KG}(U\uparrow_H^G, V), \beta \mapsto \varepsilon\circ\beta\uparrow_H^G\,.$$

Proposition 27.4 (*Characterisation of relative projectivity*)

Let $H \leqslant G$. Let U be a KG-module. Then the following are equivalent.

(a) The KG-module U is relatively H-projective.

(b) If $\psi : U \longrightarrow W$ is a KG-homomorphism, $\varphi : V \twoheadrightarrow W$ is a surjective KG-homomorphism and there exists a KH-homomorphism $\alpha_H : U\downarrow_H^G \longrightarrow V\downarrow_H^G$ such that $\varphi \circ \alpha_H = \psi$ on $U\downarrow_H^G$, then there exists a KG-homomorphism $\alpha_G : U \longrightarrow V$ such that $\varphi \circ \alpha_G = \psi$ so that the diagram on the right commutes.

$$\begin{tikzcd} & U \arrow[d, "\psi"] \arrow[dl, dashed, "\exists\,\alpha_G"'] \\ V \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft"] & W \end{tikzcd}$$

(c) Whenever $\varphi : V \twoheadrightarrow U$ is a surjective KG-homomorphism such that the restriction $\varphi : V\downarrow_H^G \longrightarrow U\downarrow_H^G$ splits as KH-homomorphism, then φ splits as a KG-homomorhpism.

(d) The surjective KG-homomorphism

$$\begin{aligned} U\downarrow_H^G\uparrow_H^G = KG \otimes_{KH} U &\longrightarrow U \\ x \otimes u &\mapsto xu \end{aligned}$$

is split.

(e) The KG-module U is a direct summand of $U\downarrow_H^G\uparrow_H^G$.

(f) There exists a KG-module N such that $U \mid K\uparrow_H^G \otimes_K N$.

Proof:

(a)⇒(b): First we consider the case in which $U = T\uparrow_H^G$ is an induced module. Suppose that we have KG-homomorphisms $\psi : T\uparrow_H^G \longrightarrow W$ and $\varphi : V \twoheadrightarrow W$ as shown in the diagram shown on the left below. Suppose, moreover, that there exists a KH-homomorhpism $\alpha_H : T\uparrow_H^G\downarrow_H^G \longrightarrow V\downarrow_H^G$ such that $\psi = \varphi \circ \alpha_H$, that is, the diagram on the right below commutes:

$$\begin{tikzcd} & T\uparrow_H^G \arrow[d, "\psi"] \\ V \arrow[r, two heads, "\varphi"'] & W \end{tikzcd} \qquad \begin{tikzcd} & T\uparrow_H^G\downarrow_H^G \arrow[d, "\psi"] \arrow[dl, dashed, "\exists\,\alpha_H"'] \\ V\downarrow_H^G \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft"] & W\downarrow_H^G \end{tikzcd}$$

Let $\mu : T \longrightarrow T\uparrow_H^G\downarrow_H^G$ and $\varepsilon : V\downarrow_H^G\uparrow_H^G \longrightarrow V$ be the unit and the counit of the adjunction of Res_H^G and Ind_H^G as defined in Notation 27.3, so μ is an injective KH-homomorphism and ε is a surjective KG-homomorphism. Then, precomposing with μ, we obtain that the following triangle of KH-modules and KH-homomorphisms commutes:

$$\begin{tikzcd} & T \arrow[d, "\psi\circ\mu"] \arrow[dl, "\alpha_H\circ\mu"'] \\ V\downarrow_H^G \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft"] & W\downarrow_H^G \end{tikzcd}$$

By the naturality of Φ and Ψ from Notation 27.3, since $\varphi : V \longrightarrow W$ is a KG-homomorphism, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{KH}(T, V\downarrow^G_H) \arrow[r, "\Psi_{T,V}", "\cong"'] \arrow[d, "\varphi_*"'] & \mathrm{Hom}_{KG}(T\uparrow^G_H, V) \arrow[d, "\varphi_*"] \\
\mathrm{Hom}_{KH}(T, W\downarrow^G_H) \arrow[r, "\cong", "\Psi_{T,W}"'] & \mathrm{Hom}_{KG}(T\uparrow^G_H, W)
\end{tikzcd}$$

In other words,

$$\Psi(\varphi \circ (\alpha_H \circ \mu)) = \varphi \circ (\Psi(\alpha_H \circ \mu))\,.$$

By the commutativity of the previous triangle, the left hand side of this equation is equal to $\Psi(\psi \circ \mu) = \Psi(\Phi(\psi)) = \psi$ since Ψ and Φ are inverse to one another. Thus

$$\psi = \varphi \circ \varepsilon \circ ((\alpha_H \circ \mu)\uparrow^G_H)$$

and so the triangle of KG-homomorphisms

$$\begin{tikzcd}
 & T\uparrow^G_H \arrow[dl, "\varepsilon\circ((\alpha_H\circ\mu)\uparrow^G_H)"'] \arrow[d, "\psi"] \\
V \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft" near end] & W
\end{tikzcd}$$

commutes, proving the implication for $U = T\uparrow^G_H$ an induced module.

Now let U be any direct summand of $T\uparrow^G_H$. Let $U \xrightarrow{\iota} T\uparrow^G_H \xrightarrow{\pi} U$ denote the canonical inclusion and projection. Suppose that there is a KH-homomorphism $\alpha_H : U\downarrow^G_H \longrightarrow V\downarrow^G_H$ such that the diagram

$$\begin{tikzcd}
 & U\downarrow^G_H \arrow[dl, "\exists\,\alpha_H"'] \arrow[d, "\psi"] \\
V\downarrow^G_H \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft" near end] & W\downarrow^G_H
\end{tikzcd}$$

commutes, i.e. $\varphi \circ \alpha_H = \psi$ on $U\downarrow^G_H$. Then we consider the following diagrams:

$$\begin{tikzcd}
 & T\uparrow^G_H \arrow[d, "\psi\circ\pi"] \\
V \arrow[r, two heads, "\varphi"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T\uparrow^G_H\downarrow^G_H \arrow[dl, "\alpha_H\circ\pi"'] \arrow[d, "\psi\circ\pi"] \\
V\downarrow^G_H \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft" near end] & W\downarrow^G_H
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T\uparrow^G_H \arrow[dl, "\alpha_G"'] \arrow[d, "\psi\circ\pi"] \\
V\downarrow^G_H \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft" near end] & W
\end{tikzcd}$$

The middle diagram of KH-homomorphisms commutes by definition of α_H, and hence by the first part there is a KG-homomorphism $\alpha_G : T\uparrow^G_H \longrightarrow V$ such that $\varphi \circ \alpha_G = \psi \circ \pi$, so the third diagram of KG-homomorhpisms also commutes.

Now $\varphi \circ \alpha_G \circ \iota = \psi \circ \pi \circ \iota = \psi$, so the triangle

$$\begin{tikzcd}
 & U \arrow[dl, "\alpha_G\circ\iota"'] \arrow[d, "\psi"] \\
V \arrow[r, two heads, "\varphi"'] \arrow[ur, phantom, "\circlearrowleft" near end] & W
\end{tikzcd}$$

commutes, as required.

(b)$\Rightarrow$(c): Let $\varphi : V \twoheadrightarrow U$ be a surjective KG-homomorphism which is split as a KH-homomorphism, and let α_H be a KH-section for φ. Thus, we have the following commutative diagram of KH-modules:

$$\begin{tikzcd} & U\downarrow_H^G \arrow[dl, "\alpha_H"'] \arrow[d, "\mathrm{Id}_U"] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] \\ V\downarrow_H^G \arrow[r, two heads, "\varphi"'] & U\downarrow_H^G \end{tikzcd}$$

Then assuming (b) is true, there exists a KG-homomorphism $\alpha_G : U \longrightarrow V$ such that $\varphi \circ \alpha_G = \mathrm{Id}_U$. In particular, α_G is a KG-section for φ.

(c)⇒(d): Since $\mu : U \longrightarrow U\downarrow_H^G\uparrow_H^G$ is a KH-section for $\varepsilon : U\downarrow_H^G\uparrow_H^G \longrightarrow U$ (see Notation 27.3), applying condition (c) yields that ε splits as a KG-homomorhpism, and hence (d) holds.

(d)⇒(e): immediate.

(e)⇒(f): Recall that by Proposition 17.11 we have $K\uparrow_H^G \otimes_K N \cong (K \otimes_K N\downarrow_H^G)\uparrow_H^G \cong N\downarrow_H^G\uparrow_H^G$. Thus, setting $N := U$ yields the claim.

(f)⇒(a): straightforward from the fact that $K\uparrow_H^G \otimes_K N \cong N\downarrow_H^G\uparrow_H^G$ seen above. ■

Exercise 27.5

Let $H \leqslant J \leqslant G$. Let U be a KG-module and let V be a KJ-module. Prove the following statements.

(a) If U is H-projective then U is J-projective.

(b) If U is a direct summand of $V\uparrow_J^G$ and V is H-projective, then U is H-projective.

(c) For any $g \in G$, U is H-projective if and only if ${}^gU$ is ${}^gH$-projective.

(d) Using part (f) of Proposition 27.4, prove that if U is H-projective and W is any KG-module, then $U \otimes_K W$ is H-projective.

Projectivity relative to a subgroup can be generalised as follows to projectivity relative to a KG-module:

Remark 27.6 (*Projectivity relative to KG-modules*)

(a) Let V be a KG-module. A KG-module M is termed *projective relative to the module* V or *relatively* V*-projective*, or simply V*-projective* if there exists a KG-module N such that M is isomorphic to a direct summand of $V \otimes_K N$, i.e. $M \mid V \otimes_K N$.
We let $\mathrm{Proj}(V)$ denote the class of all V-projective KG-modules.

(b) Proposition 27.4(f) shows that projectivity relative to a subgroup $H \leqslant G$ is in fact projectivity relative to the KG-module $V := K\uparrow_H^G$.

Note that the concept of projectivity relative to a subgroup is proper to the group algebra, but the concept of projectivity relative to a module is not and makes sense in general over algebras/rings.

The following exercise provides us with some elementary properties of projectivity relative to a module, which also hold for projectivity relative to a subgroup, by part (b) of the remark.

Exercise 27.7

Assume K is a field of characteristc $p > 0$ and let A, B, C, U, V be KG-modules. Prove that:

(a) Any direct summand of a V-projective KG-module is V-projective;

(b) If $U \in \mathrm{Proj}(V)$, then $\mathrm{Proj}(U) \subseteq \mathrm{Proj}(V)$;

(c) If $p \nmid \dim_K(V)$ then any KG-module is V-projective;

(d) $\mathrm{Proj}(V) = \mathrm{Proj}(V^*)$;

(e) $\mathrm{Proj}(U \oplus V) = \mathrm{Proj}(U) \oplus \mathrm{Proj}(V)$;

(f) $\mathrm{Proj}(U) \cap \mathrm{Proj}(V) = \mathrm{Proj}(U \otimes_K V)$;

(g) $\mathrm{Proj}(\bigoplus_{j=1}^{n} V) = \mathrm{Proj}(V) = \mathrm{Proj}(\bigotimes_{j=1}^{m} V)\ \ \forall\, m, n \in \mathbb{Z}_{>0}$;

(h) $C \cong A \oplus B$ is V-projective if and only if both A and B are V-projective;

(i) $\mathrm{Proj}(V) = \mathrm{Proj}(V^* \otimes_K V)$.

Hint: you may want to use Lemma 13.8 and Exercise 4(c) on Sheet 3. Proceed in the given order.

After this small parenthesis on projectivity relative to modules, we come back to projectivity relative to subgroups. We investigate further what information this concept brings to the understanding of indecomposable KG-modules in general.

Next we see that any indecomposable KG-module can be seen as a relatively projective module with respect to some subgroup of G.

Theorem 27.8

Let $H \leqslant G$.

(a) If $|G : H|$ is invertible in K, then every KG-module is H-projective.

(b) In particular, if K is a field of characteristic $p > 0$ and H contains a Sylow p-subgroup of G, then every KG-module is H-projective.

Proof: (a) Let V be a KG-module. To prove that V is H-projective, we prove that V satisfies Theorem 27.4(c). So let $\varphi : U \twoheadrightarrow V$ be a surjective KG-homomorphism which splits as a KH-homomorphism. We need to prove that φ splits as a KG-homomorphism.
So let $\sigma : V \longrightarrow U$ be a KH-linear section for φ and set

$$\begin{array}{rccc} \tilde{\sigma}: & V & \longrightarrow & U \\ & v & \mapsto & \frac{1}{|G:H|}\sum_{g\in[G/H]} g^{-1}\sigma(gv). \end{array}$$

We may divide by $|G : H|$ since $|G : H| \in K^\times$ and clearly $\tilde{\sigma}$ is well-defined. Now, if $g' \in G$ and $v \in V$, then

$$\tilde{\sigma}(g'v) = \frac{1}{|G:H|}\sum_{g\in[G/H]} g^{-1}\sigma(gg'v) = g'\frac{1}{|G:H|}\sum_{g\in[G/H]} (gg')^{-1}\sigma(gg'v) = g'\tilde{\sigma}(v)$$

and

$$\varphi\widetilde{\sigma}(v) = \frac{1}{|G:H|}\sum_{g\in G}\varphi\left(g^{-1}\sigma(gv)\right) \overset{\varphi\,KG\text{-lin.}}{=} \frac{1}{|G:H|}\sum_{g\in G}g^{-1}\varphi\sigma(gv) = \frac{1}{|G:H|}\sum_{g\in G}g^{-1}gv = v$$

where the last-but-one equality holds because $\varphi\sigma = \mathrm{Id}_V$. Thus $\widetilde{\sigma}$ is a KG-linear section for φ.

(b) follows immediately from (a). Indeed, if $P \in \mathrm{Syl}_p(G)$ and $H \supseteq P$, then $p \nmid |G:H|$, so $|G:H| \in K^\times$. ■

Considering the case $H = \{1\}$ shows that the previous Theorem is in some sense a generalisation of Maschke's Theorem (Theorem 11.1).

Remark 27.9

Assume that K is a field of characteristic $p > 0$ and $H = \{1\}$ is the trivial subgroup. If H contains a Sylow p-subgroup of G then the Sylow p-subgroups of G are trivial, so $p \nmid |G|$. The theorem then says that all KG-modules are $\{1\}$-projective and hence projective.

We know this already, however! If $p \nmid |G|$ then KG is semisimple by Maschke's Theorem (Theorem 11.1), and so all KG-modules are projective by Example 13(d).

Corollary 27.10

Let $H \leqslant G$ and suppose that $|G:H|$ is invertible in K. Then a KG-module U is projective if and only if $U\!\downarrow_H^G$ is projective.

Again, this holds in particular if K is a field of characteristic $p \geqslant 0$ and H contains a Sylow p-subgroup of G.

Proof: The necessary condition is given by Proposition 23.1(b). To prove the sufficient condition, suppose that $U\!\downarrow_H^G$ is projective. Then, on the one hand,

$$U\!\downarrow_H^G \mid (KH)^n \qquad \text{for some } n \in \mathbb{Z}_{>0}\,.$$

On the other hand, U is H-projective by Theorem 27.8, and it follows from Proposition 27.4(e) that

$$U \mid U\!\downarrow_H^G\uparrow_H^G\,.$$

Hence

$$U \mid U\!\downarrow_H^G\uparrow_H^G \mid (KH)^n\uparrow_H^G \cong (KG)^n\,,$$

so U is projective. ■