**Representation Theory — Exercise Sheet 6** **RPTU Kaiserslautern-Landau**

**Jun.-Prof. Dr. Caroline Lassueur** **FB Mathematik**

**Marie Roth**

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Throughout, $G$ denotes a finite group, $p$ is a prime number and $(F, \mathcal{O}, k)$ is a splitting $p$-modular system for $G$ and its subgroups. Furthermore, all modules considered are assumed to be *left* modules and finitely generated.

**Exercise 1.**
Let $\sigma : G \longrightarrow H$ be an isomorphism of groups. If $\psi$ is a class function defined on $G$ (resp. on $G_{p'}$), we define

$$\psi^\sigma(x) := \psi(x^{\sigma^{-1}}) \qquad \forall\, x \in H \text{ (resp. } \forall\, x \in H_{p'})$$

Prove that $\psi$ is a Brauer character of $G$ if and only if $\psi^\sigma$ is a Brauer character of $H$. Prove that moreover $d_{\chi\varphi} = d_{\chi^\sigma\varphi^\sigma}$ for every $\chi \in \mathrm{Irr}(G)$ and every $\varphi \in \mathrm{IBr}(G)$.

**Exercise 2.**
Let $U$ be a $kG$-module and let $P$ be a PIM of $kG$. Prove that

$$\dim_k \mathrm{Hom}_{kG}(P, U) = \frac{1}{|G|} \sum_{g \in G_{p'}} \varphi_P(g^{-1})\varphi_U(g)\,.$$

**Exercise 3.**
Let $G := \mathfrak{A}_5$ be the alternating group on 5 letters. Calculate the Brauer character table, the Cartan matrix and the decomposition matrix of $G$ for $p = 3$.

[Hints. (1.) Use the ordinary character table of $\mathfrak{A}_5$ and reduction modulo $p$. (2.) Use the fact that a simple group does not have any irreducible Brauer character of degree 2.]

**Exercise 4.**
Let $A$ be a ring and let $A = A_1 \oplus \cdots \oplus A_r$ ($r \in \mathbb{Z}_{\geq 1}$) be the block decomposition of $A$ and let $M$ be an arbitrary $A$-module. Prove that $M$ admits a unique direct sum decomposition $M = M_1 \oplus \cdots \oplus M_r$ where for each $1 \leq i \leq r$ the summand $M_i$ belongs to the block $A_i$ of $A$. Deduce that every indecomposable $A$-module lies in a uniquely determined block of $A$.

**Exercise 5.**
Let $B \in \mathrm{Bl}_p(\mathcal{O}G)$. Prove that an $\mathcal{O}G$-module $M$ belongs to $B$ if and only if $M/\mathfrak{p}M$ belongs to the image $\overline{B} \in \mathrm{Bl}_p(kG)$ of $B$.

**Exercise** 6.
Prove Proposition 40.3.

[Hints for (a): Let $E$ be a defect group of $B := b^G$. Then $B$ is a direct summand of $V\Delta(E)G \times G$ for some $\Delta(E)$-module $V$. Consider $V\Delta(E)G \times G \downarrow^{G\times G}_{H\times H}$.
Hints for (b): Part (b) essentially follows from the definitions.
Hints for (c): Justify that it is enough to prove that $b$ occurs precisely once in a decomposition of $kG\downarrow^{G\times G}_{H\times H}$ into indecomposable modules. Use the remark before the proposition.]

**Exercise** 7.
Verify that the Brauer correspondence is a particular case of the Green correspondence.