**Representation Theory — Exercise Sheet 5** | **TU Kaiserslautern**

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Throughout, $G$ denotes a finite group.

**Exercise** 1.
Let $\mathcal{O}$ be a complete discrete valuation ring with $k := \mathcal{O}/J(\mathcal{O})$. Let $A$ be a finitely generated $\mathcal{O}$-algebra and let $e \in A$ be an idempotent element. Prove that:

(a) $J(eAe) = eJ(A)e$;

(b) if $M$ is an $A$-module, then $\operatorname{Hom}_A(Ae, M) \cong eM$ as $\operatorname{End}_A(M)$-modules;

(c) $e$ is primitive if and only if the left ideal $Ae$ is indecomposable if and only if $e$ and $0$ are the only idempotents of $eAe$;

(d) if $A$ is a finite-dimensional $k$-algebra, then $e$ is primitive if and only if $eAe$ is a local ring (in which case $eJ(A)e$ is the unique maximal ideal of $eAe$).

**Exercise** 2.
Let $\mathcal{O}$ be a complete discrete valuation ring and write $\mathfrak{p} := J(\mathcal{O})$. Let $A$ be a finitely generated $\mathcal{O}$-algebra. Set $\overline{A} := A/\mathfrak{p}A$ and for $a \in A$ write $\overline{a} := a + \mathfrak{p}A$. Prove that:

(a) For every idempotent $x \in \overline{A}$, there exists an idempotent $e \in A$ such that $\overline{e} = x$.

(b) We have $A^\times = \{a \in A \mid \overline{a} \in \overline{A}^\times\}$.

(c) If $e_1, e_2 \in A$ are idempotents such that $\overline{e}_1 = \overline{e}_2$ then there is a unit $u \in A^\times$ such that $e_1 = ue_2u^{-1}$ .

(d) The quotient morphism $A \to \overline{A}$ induces a bijection between the central idempotents of $A$ and the central idempotents of $\overline{A}$.

From now on, we assume that $(F, \mathcal{O}, k)$ is a splitting $p$-modular system for $G$ and its subgroups. For $K \in \{F, \mathcal{O}, k\}$ all $KG$-modules considered are assumed to be *left* modules and free of finite rank over $K$.

**Exercise** 3.
Prove that if $L$ is an indecomposable $p$-permutation $\mathcal{O}G$-lattice, then $L/\mathfrak{p}L$ is an indecomposable $p$-permutation $kG$-module.
[Optional exercise: if $Q \in \operatorname{vtx}(L)$, then $Q \in \operatorname{vtx}(M)$.]

**Exercise 4.**
Assume $K \in \{\mathcal{O}, k\}$. Recall that a *primitive decomposition* of an idempotent element $e \in KG$ is a decomposition of $e$ of the form $e = \sum_{i \in I} i$ where $I$ is a set of pairwise orthogonal primitive idempotents of $KG$. Prove that a decomposition of a $KG$-module $M$ into a direct sum of indecomposable summands amounts to choosing a primitive decomposition of $\mathrm{Id}_M \in \mathrm{End}_{KG}(M)$.

**Exercise 5.**
Let $U, V, W$ be $kG$-modules. Prove that the following assertions.

(a) If $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is a short exact sequence of $kG$-modules, then

$$\varphi_V = \varphi_U + \varphi_W \,.$$

(b) If the composition factors of $U$ are $S_1, \ldots, S_m$ ($m \in \mathbb{Z}_{\geq 1}$) with multiplicities $n_1, \ldots, n_m$ respectively, then

$$\varphi_U = n_1 \varphi_{S_1} + \ldots + n_m \varphi_{S_m} \,.$$

In particular, if two $kG$-modules have isomorphic composition factors, counting multiplicities, then they have the same Brauer character.

(c) We have $\varphi_{U \oplus V} = \varphi_U + \varphi_V$ and $\varphi_{U \otimes_k V} = \varphi_U \cdot \varphi_V$.

**Exercise 6.**
Prove that two $kG$-modules afford the same Brauer character if and only if they have isomorphic composition factors (including multiplicities).

**Exercise 7.**
Let $\varphi, \lambda \in \mathrm{IBr}_p(G)$ and assume that $\lambda$ is linear. Prove that $\lambda\varphi \in \mathrm{IBr}_p(G)$ and

$$\lambda(\Phi_\varphi)|_{G_{p'}} = (\Phi_{\lambda\varphi})|_{G_{p'}} \,.$$

**Exercise 8.**
Let $G$ be a finite group and let $\rho_{reg}$ denote the regular $F$-character of $G$. Prove that:

$$\rho_{\mathrm{reg}} = \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)\Phi_\varphi \quad \text{and} \quad (\rho_{\mathrm{reg}})|_{G_{p'}} = \sum_{\varphi \in \mathrm{IBr}_p(G)} \Phi_\varphi(1)\varphi \,.$$