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Throughout, G denotes a finite group, and for simplicity, K is a field. All KG-modules considered are assumed to be finitely generated.

Exercise 1.
Prove that $\mathrm{Coind}_{\{1\}}^G(K) \cong (KG)^*$ as KG-modules by defining an explicit KG-isomorphism.

Exercise 2.
Let U, V, W be KG-modules. Prove that there are isomorphisms of KG-modules:

(i) $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(U, V^* \otimes_K W)$; and

(ii) $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W) \cong \mathrm{Hom}_{KG}(U, \mathrm{Hom}_K(V, W))$.

Exercise 3.

(a) Let $H, L \leq G$. Prove that the set of (H, L)-double cosets is in bijection with the set of orbits $H\backslash(G/L)$, and also with the set of orbits $(H\backslash G)/L$ under the mappings

$$HgL \mapsto H(gL) \in H\backslash(G/L)$$

$$HgL \mapsto (Hg)L \in (H\backslash G)/L.$$

This justifies the notation $H\backslash G/L$ for the set of (H, L)-double cosets.

(b) Let $G = S_3$. Consider $H = L := S_2 = \{\mathrm{Id}, (1\ 2)\}$ as a subgroup of S_3. Prove that

$$[S_2\backslash S_3/S_2] = \{\mathrm{Id}, (1\ 2\ 3)\}$$

while

$$S_2\backslash S_3/S_2 = \{\, \{\mathrm{Id}, (1\ 2)\}, \{(1\ 2\ 3), (1\ 3\ 2), (1\ 3), (2\ 3)\}\,\}\,.$$

Exercise 4.
If $H \leq G$ and P is a projective KH-module, then $P\!\uparrow_H^G$ is a projective KG-module.

Exercise 5.
Let $H, L \leq G$, let M be a KL-module and let N be a KH-module. Use the Mackey formula to prove that:

(a) $M\uparrow_L^G \otimes_K N\uparrow_H^G \cong \bigoplus_{g\in[H\backslash G/L]} ({}^gM\downarrow^{{}^gL}_{H\cap {}^gL} \otimes_K N\downarrow^H_{H\cap {}^gL})\uparrow^G_{H\cap {}^gL}$;

(b) $\mathrm{Hom}_K(M\uparrow_L^G, N\uparrow_H^G) \cong \bigoplus_{g\in[H\backslash G/L]} (\mathrm{Hom}_K({}^gM\downarrow^{{}^gL}_{H\cap {}^gL}, N\downarrow^H_{H\cap {}^gL}))\uparrow^G_{H\cap {}^gL}$.

Exercise 6.
Let M be a KG-module.

(a) Prove that the following KG-submodules of M are equal:

   (1) $\mathrm{soc}(M)$;
   (2) the largest semisimple KG-submodule of M;
   (3) $\{m \in M \mid J(KG) \cdot m = 0\}$.

   [Hint: In words, $\{m \in M \mid J(KG) \cdot m = 0\}$ is the largest submodule of M annihilated by $J(KG)$.]

(b) Prove that $\mathrm{rad}^n(M) = J(KG)^n \cdot M$ and $\mathrm{soc}^n(M) = \{m \in M \mid J(KG)^n \cdot m = 0\}$ for every $n \in \mathbb{Z}_{\geq 2}$.

(c) Check that we have chains of KG-submodules of M given by:
$\cdots \subseteq \mathrm{rad}^3(M) \subseteq \mathrm{rad}^2(M) \subseteq \mathrm{rad}(M) \subseteq M$ and $0 \subseteq \mathrm{soc}(M) \subseteq \mathrm{soc}^2(M) \subseteq \mathrm{soc}^3(M) \subseteq \cdots$

Exercise 7.
Let M and N be KG-modules. Prove the following assertions.

(a) For every $n \in \mathbb{Z}_{\geq 1}$, we have

$$\mathrm{rad}^n(M \oplus N) \cong \mathrm{rad}^n(M) \oplus \mathrm{rad}^n(N) \qquad \text{and} \qquad \mathrm{soc}^n(M \oplus N) \cong \mathrm{soc}^n(M) \oplus \mathrm{soc}^n(N).$$

(b) The radical series of M is the fastest descending series of KG-submodules of M with semisimple quotients, and the socle series of M is the fastest ascending series of M with semisimple quotients. The two series terminate, and if r and n are the least integers for which $\mathrm{rad}^r(M) = 0$ and $\mathrm{soc}^n(U) = M$ then $r = n$.

Exercise 8.
Let S be a simple KG-module and let P_S denote the corresponding PIM (i.e. $P_S/\,\mathrm{rad}(P_S) \cong S$). Let M be an arbitrary KG-module. Prove the following assertions.

(a) If T is a simple KG-module then

$$\dim_K \mathrm{Hom}_{KG}(P_S, T) = \begin{cases} \dim_K \mathrm{End}_{KG}(S) & \text{if } S \cong T, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The multiplicity of S as a composition factor of M is

$$\dim_K \mathrm{Hom}_{KG}(P_S, M)/\dim_K \mathrm{End}_{KG}(S).$$