Jun.-Prof. Dr. Caroline Lassueur | **FB Mathematik**
Marie Roth

Due date: **Tuesday, 22nd of November 2022, 2 p.m.** WS 2022/23

Throughout, all rings are assumed to be *associative rings with* a one, modules are assumed to be *left* modules and finitely generated.

Exercise 1.
Let $G := C_2 \times C_2$ be the Klein-four group and let $K = \overline{K}$ be an algebraically closed field of characteristic 2.

(i) Prove that $KG \cong K[X,Y]/(X^2,Y^2)$ as K-algebras.
(Note: $K[X,Y]$ stands for the commutative polynomial K-algebra in the variables X and Y, i.e. $XY = YX$ in $K[X,Y]$.)

(ii) Compute $J\left(K[X,Y]/(X^2,Y^2)\right)$, $\left|\mathrm{Irr}\left(K[X,Y]/(X^2,Y^2)\right)\right|$, and describe all simple KG-modules.
[Hint: Do not forget that you can consider K-dimensions!]

Exercise 2.
Let K be a field and let $A \neq 0$ be a finite-dimensional K-algebra. The aim of this exercise is to prove that $J(A)$ is the unique maximal nilpotent left ideal of A and $J(Z(A)) = J(A) \cap Z(A)$.
Proceed as follows:

(a) Prove that there exists $n \in \mathbb{Z}_{>0}$ such that $J(A)^n = J(A)^{n+1}$.
[Hint: consider dimensions.]

(b) Apply Nakayama's Lemma to deduce that $J(A)^n = 0$ and conclude that $J(A)$ is nilpotent.

(c) Prove that if I is an arbitrary nilpotent left ideal of A, then $I \subseteq J(A)$.
[Hint: here you should see $J(A)$ as the intersection of the annihilators of the simple A-modules.]

(d) Use the nilpotency of the Jacobson radical to prove that $J(Z(A)) = J(A) \cap Z(A)$.

Exercise 3.
The aim of this exercise is to prove that if K is a field of positive characteristic p and G is a p-group, then $I(KG) = J(KG)$. Proceed as indicated below.

(a) Recall that an ideal I of a ring R is called a **nil ideal** if each element of I is nilpotent. Accept the following result: if I is a nil left ideal in a left Artinian ring R then I is nilpotent.

(b) Prove that $g - 1$ is a nilpotent element for each $g \in G \setminus \{1\}$ and deduce that $I(KG)$ is a nil ideal of KG.

(c) Deduce from (a) and (b) that $I(KG) \subseteq J(KG)$ using Exercise 2.

(d) Conclude that $I(KG) = J(KG)$ using Proposition-Definition 10.7.

Exercise 4 (**Proof of the Converse of Maschke's Theorem for K a splitting field for KG.**).
Assume K is a field of positive characteristic p with $p \mid |G|$ and is a splitting field for KG. Set $T := \langle \sum_{g \in G} g \rangle_K$.

(a) Prove that we have a series of KG-submodules given by $KG^\circ \supsetneq I(KG) \supseteq T \supsetneq 0$.

(b) Deduce that KG° has at least two composition factors isomorphic to the trivial module.

(c) Deduce that KG is not a semisimple K-algebra using Theorem 8.2.

Exercise 5.
Let $\mathcal{O}$ be a local commutative ring with unique maximal ideal $\mathfrak{p} := J(\mathcal{O})$ and residue field $k := \mathcal{O}/J(\mathcal{O})$.

(a) Let M, N be finitely generated free $\mathcal{O}$-modules.

  (i) Let $f : M \longrightarrow N$ be an $\mathcal{O}$-linear map and $\overline{f} : \overline{M} \longrightarrow \overline{N}$ be its reduction modulo $\mathfrak{p}$. Prove that if $\overline{f}$ is surjective (resp. an isomorphism), then f is surjective (resp. an isomorphism).

  (ii) Prove that if elements $x_1, \ldots, x_n \in M$ ($n \in \mathbb{Z}_{\geq 1}$) are such that their images $\overline{x}_1, \ldots \overline{x}_2 \in \overline{M}$ form a k-basis of $\overline{M}$, then $\{x_1, \ldots, x_n\}$ is an $\mathcal{O}$-basis of M.
  In particular, $\dim_k(\overline{M}) = \mathrm{rk}_{\mathcal{O}}(M)$.

[Hint: Use Nakayama's Lemma.]

(b) Any direct summand of a finitely generated free $\mathcal{O}$-module is free.

(c) Prove that if M is a finitely generated $\mathcal{O}$-module, then the following conditions are equivalent:

  (i) M is projective;

  (ii) M is free.

Exercise 6.
Let $(F, \mathcal{O}, k)$ be a p-modular system and set $\mathfrak{p} := J(\mathcal{O})$.

(a) Given an $\mathcal{O}G$-lattice L, verify that:

  · setting $L^F := F \otimes_{\mathcal{O}} L$ defines an FG-module, and

  · reduction modulo $\mathfrak{p}$ of L, i.e. $\overline{L} := L/\mathfrak{p}L \cong k \otimes_{\mathcal{O}} L$, defines a kG-module.

(b) Let V be a finitely generated FG-module and let $\{v_1, \ldots, v_n\}$ be an F-basis of V. Prove that $L := \mathcal{O}Gv_1 + \cdots + \mathcal{O}Gv_n \subseteq V$ is an $\mathcal{O}$-form of V.

Exercise 7.
Let $(F, \mathcal{O}, k)$ be a p-modular system. Prove the following assertions.

(a) If $K \in \{F, \mathcal{O}, k\}$ and M is a finitely generated KG-lattice, then Tr_M is a KG-homomorphism and $\mathrm{Tr}_M \circ \theta^{-1}_{M,M}$ coincides with the ordinary trace of matrices.

(b) If M is a kG-module, then:

  (i) $M \mid M \otimes_k M^* \otimes_k M$, and

  (ii) $M \oplus M \mid M \otimes_k M^* \otimes_k M$ provided $\mathrm{char}(k) \mid \dim_k(M)$. [This is more challenging!]