Representation Theory — Exercise Sheet 1 **TU Kaiserslautern**
Jun.-Prof. Dr. Caroline Lassueur **FB Mathematik**
Marie Roth
Due date: **Tuesday, 8th of November 2022, 2 p.m.** WS 2022/23

Throughout, R denotes a ring, and, unless otherwise stated, all rings are assumed to be *associative rings with* 1, and modules are assumed to be *left* modules.

Exercise 1.
Prove that if $(R, +, \cdot)$ is a ring, then $R^\circ := R$ itself may be seen as an R-module via left multiplication in R, i.e. where the external composition law is given by

$$R \times R^\circ \longrightarrow R^\circ, (r, m) \mapsto r \cdot m\,.$$

We call R° the **regular** R-module.
Prove that:

(a) the R-submodules of R° are prescisely the left ideals of R;

(b) $I \triangleleft R$ is a maximal left ideal of $R \Leftrightarrow R^\circ/I$ is a simple R-module, and $I \triangleleft R$ is a minimal left ideal of $R \Leftrightarrow I$ is simple when regarded as an R-submodule of R°.

Exercise 2.
Give a concrete example of an R-module which is indecomposable but not simple.

Exercise 3.
Prove Part (iii) of Fitting's Lemma.

Exercise 4.
Let p be a prime number and let $R := \{\frac{a}{b} \in \mathbb{Q} \,\big|\, p \nmid b\}$. Determine $R \setminus R^\times$ and deduce that R is local.

Exercise 5.

(a) Prove that any simple R-module may be seen as a simple $R/J(R)$-module.

(b) Conversely, prove that any simple $R/J(R)$-module may be seen as a simple R-module. [Hint: use a change of the base ring via the canonical morphism $R \longrightarrow R/J(R)$.]

(c) Deduce that R and $R/J(R)$ have the same simple modules.

Exercise 6.

(a) Prove that any submodule and any quotient of a semisimple module is again semisimple.

(b) Let K be a field and let A be the K-algebra $\left\{\begin{pmatrix} a_1 & a \\ 0 & a_1 \end{pmatrix} \mid a_1, a \in K\right\}$. Consider the A-module $V := K^2$, where A acts by left matrix multiplication. Prove that:

   (1) $\{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in K\}$ is a simple A-submodule of V; but

   (2) V is not semisimple.

(b) Prove that $J(\mathbb{Z}) = 0$ and find an example of a $\mathbb{Z}$-module which is not semisimple.

Exercise 7.
Let R be a semisimple ring. Prove the following statements.

(a) Every non-zero left ideal I of R is generated by an **idempotent** of R, in other words $\exists\, e \in R$ such that $e^2 = e$ and $I = Re$.
[Hint: choose a complement I' for I, so that $R^\circ = I \oplus I'$ and write $1 = e + e'$ with $e \in I$ and $e' \in I'$. Prove that $I = Re$.]

(b) If I is a non-zero left ideal of R, then every morphism in $\mathrm{Hom}_R(I, R^\circ)$ is given by right multiplication with an element of R.

(c) If $e \in R$ is an idempotent, then $\mathrm{End}_R(Re) \cong (eRe)^{\mathrm{op}}$ (the opposite ring) as rings via the map $f \mapsto ef(e)e$. In particular $\mathrm{End}_R(R^\circ) \cong R^{\mathrm{op}}$ via $f \mapsto f(1)$.

(d) A left ideal Re generated by an idempotent e of R is minimal (i.e. simple as an R-module) if and only if eRe is a division ring.
[Hint: Use Schur's Lemma.]

(e) Every simple left R-module is isomorphic to a minimal left ideal in R, i.e. a simple R-submodule of R°.