# Appendix 1: Background Material Module Theory

This appendix provides you with a short recap of the notions of the theory of modules, which we will assume as known for this lecture. We quickly review elementary definitions and constructions such as quotients, direct sum, direct products, tensor products and exact sequences, where we emphasise the approach via universal properties.

**Notation**: throughout this appendix we let $R$ and $S$ denote rings, and unless otherwise specified, all rings are assumed to be *unital* and *associative*.

Most of the results are stated without proof, as they have been studied in the B.Sc. lecture *Commutative Algebra*. As further reference I recommend for example:

## A Modules, submodules, morphisms

**Definition A.1 (*Left $R$-module, right $R$-module, $(R,S)$-bimodule*)**

(a) A **left $R$-module** is an ordered triple $(M, +, \cdot)$, where $(M, +)$ is an abelian group and $\cdot : R \times M \longrightarrow M, (r, m) \mapsto r \cdot m$ is a binary operation such that the map

$$\begin{array}{rccl} \lambda : & R & \longrightarrow & \mathrm{End}(M) \\ & r & \mapsto & \lambda(r) := \lambda_r : M \longrightarrow M, m \mapsto r \cdot m \end{array}$$

is a ring homomorphism. The operation $\cdot$ is called a **scalar multiplication** or an **external composition law**.

(b) A **right $R$-module** is defined analogously using a scalar multiplication $\cdot : M \times R \longrightarrow M$, $(m, r) \mapsto m \cdot r$ on the right-hand side.

(c) An **$(R,S)$-bimodule** is an abelian group $(M, +)$ which is both a left $R$-module and a right $S$-module, and which satisfies the axiom

$$r \cdot (m \cdot s) = (r \cdot m) \cdot s \qquad \forall\, r \in R, \forall\, s \in S, \forall\, m \in M\,.$$

**Convention**: Unless otherwise stated, in this lecture we always work with left modules. When no confusion is to be made, we will simply write "$R$-module" to mean "left $R$-module", denote $R$-modules by their underlying sets and write $rm$ instead of $r \cdot m$. Definitions for right modules and bimodules are similar to those for left modules, hence in the sequel we omit them.

**Definition A.2 (*R-submodule*)**

An **$R$-submodule** of an $R$-module $M$ is a subgroup $U \leqslant M$ such that $r \cdot u \in U \ \forall\ r \in R, \forall\ u \in U$.

**Definition A.3 (*Morphisms*)**

A **(homo)morphism** of $R$-modules (or an **$R$-linear map**, or an **$R$-homomorphism**) is a map of $R$-modules $\varphi : M \longrightarrow N$ such that:

(i) $\varphi$ is a group homomorphism; and

(ii) $\varphi(r \cdot m) = r \cdot \varphi(m)\ \forall\ r \in R, \forall\ m \in M$.

Furthermore:

- An injective (resp. surjective) morphism of $R$-modules is sometimes called a **monomorphism** (resp. an **epimorphism**) and we often denote it with a *hook arrow* "$\hookrightarrow$" (resp. a *two-head arrow* "$\twoheadrightarrow$").
- A bijective morphism of $R$-modules is called an **isomorphism** (or an **$R$-isomorphism**), and we write $M \cong N$ if there exists an $R$-isomorphism between $M$ and $N$.
- A morphism from an $R$-module to itself is called an **endomorphism** and a bijective endomorphism is called an **automorphism** .

**Notation**: We let ${}_R\mathbf{Mod}$ denote the category of left $R$-modules (with $R$-linear maps as morphisms), we let $\mathbf{Mod}_R$ denote the category of right $R$-modules (with $R$-linear maps as morphisms), and we let ${}_R\mathbf{Mod}_S$ denote the category of $(R, S)$-bimodules (with $(R, S)$-linear maps as morphisms).

**Example A.4**

(a) Exercise: Check that Definition A.1(a) is equivalent to requiring that $(M, +, \cdot)$ satisfies the following axioms:

**(M1)** $(M, +)$ is an abelian group;

**(M2)** $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$ for each $r_1, r_2 \in R$ and each $m \in M$;

**(M3)** $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$ for each $r \in R$ and all $m_1, m_2 \in M$;

**(M4)** $(rs) \cdot m = r \cdot (s \cdot m)$ for each $r, s \in R$ and all $m \in M$.

**(M5)** $1_R \cdot m = m$ for each $m \in M$.

In other words, modules over rings satisfy the same axioms as vector spaces over fields. Hence: Vector spaces over a field $K$ are $K$-modules, and conversely.

(b) Abelian groups are $\mathbb{Z}$-modules, and conversely.
Exercise: check it! What is the external composition law?

(c) If the ring $R$ is commutative, then any right module can be made into a left module, and conversely.
Exercise: check it! Where does the commutativity come into play?

(d) If $\varphi : M \longrightarrow N$ is a morphism of $R$-modules, then the kernel $\ker(\varphi) := \{m \in M \mid \varphi(m) = 0_N\}$ of $\varphi$ is an $R$-submodule of $M$ and the image $\operatorname{Im}(\varphi) := \varphi(M) = \{\varphi(m) \mid m \in M\}$ of $\varphi$ is an $R$-submodule of $N$.
If $M = N$ and $\varphi$ is invertible, then the inverse is the usual set-theoretic *inverse map* $\varphi^{-1}$ and is also an $R$-homomorphism.
Exercise: check it!

(e) **Change of the base ring:** if $\varphi : S \longrightarrow R$ is a ring homomorphism, then every $R$-module $M$ can be endowed with the structure of an $S$-module with external composition law given by

$$\cdot : S \times M \longrightarrow M, (s, m) \mapsto s \cdot m := \varphi(s) \cdot m\,.$$

Exercise: check it!

**Notation A.5**

Given $R$-modules $M$ and $N$, we set $\operatorname{Hom}_R(M, N) := \{\varphi : M \longrightarrow N \mid \varphi \text{ is an } R\text{-homomorphism}\}$. This is an abelian group for the pointwise addition of maps:

$$\begin{array}{cccc} + : & \operatorname{Hom}_R(M,N) \times \operatorname{Hom}_R(M,N) & \longrightarrow & \operatorname{Hom}_R(M,N) \\ & (\varphi,\psi) & \mapsto & \varphi + \psi : M \longrightarrow N, m \mapsto \varphi(m) + \psi(m)\,. \end{array}$$

In case $N = M$, we write $\operatorname{End}_R(M) := \operatorname{Hom}_R(M, M)$ for the set of endomorphisms of $M$ and $\operatorname{Aut}_R(M)$ for the set of automorphisms of $M$, i.e. the set of invertible endomorphisms of $M$.

**Lemma-Definition A.6 (*Quotients of modules*)**

Let $U$ be an $R$-submodule of an $R$-module $M$. The quotient group $M/U$ can be endowed with the structure of an $R$-module in a natural way via the external composition law

$$\begin{aligned} R \times M/U &\longrightarrow M/U \\ \big(r, m + U\big) &\longmapsto r \cdot m + U \end{aligned}$$

The canonical map $\pi : M \longrightarrow M/U, m \mapsto m + U$ is $R$-linear and we call it the **canonical** (or **natural**) **homomorphism**.

**Definition A.7 (*Cokernel, coimage*)**

Let $\varphi \in \operatorname{Hom}_R(M, N)$. The **cokernel** of $\varphi$ is the quotient $R$-module $\operatorname{coker}(\varphi) := N/\operatorname{Im}\varphi$, and the **coimage** of $\varphi$ is the quotient $R$-module $M/\ker\varphi$.

**Theorem A.8 (*The universal property of the quotient and the isomorphism theorems*)**

(a) **Universal property of the quotient**: Let $\varphi : M \longrightarrow N$ be a homomorphism of $R$-modules. If $U$ is an $R$-submodule of $M$ such that $U \subseteq \ker(\varphi)$, then there exists a unique $R$-module homomorphism $\overline{\varphi} : M/U \longrightarrow N$ such that $\overline{\varphi} \circ \pi = \varphi$, or in other words such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & N \\ M/U \arrow[ur, dashed, "\exists!\,\overline{\varphi}"'] & \end{tikzcd}$$

Concretely, $\overline{\varphi}(m+U) = \varphi(m)\ \forall\ m+U \in M/U$.

(b) **1st isomorphism theorem**: With the notation of (a), if $U = \ker(\varphi)$, then

$$\overline{\varphi} : M/\ker(\varphi) \longrightarrow \operatorname{Im}(\varphi)$$

is an isomorphism of $R$-modules.

(c) **2nd isomorphism theorem**: If $U_1, U_2$ are $R$-submodules of $M$, then so are $U_1 \cap U_2$ and $U_1 + U_2$, and there is an isomorphism of $R$-modules

$$(U_1 + U_2)/U_2 \cong U_1/(U_1 \cap U_2)\,.$$

(d) **3rd isomorphism theorem**: If $U_1 \subseteq U_2$ are $R$-submodules of $M$, then there is an isomorphism of $R$-modules

$$(M/U_1)\,/_{(U_2/U_1)} \cong M/U_2\,.$$

(e) **Correspondence theorem**: If $U$ is an $R$-submodule of $M$, then there is a bijection

$$\begin{array}{ccc} \{R\text{-submodules } X \text{ of } M \mid U \subseteq X\} & \longleftrightarrow & \{R\text{-submodules of } M/U\} \\ X & \mapsto & X/U \\ \pi^{-1}(Z) & \leftarrow\!\shortmid & Z\,. \end{array}$$

# B Free modules and projective modules

## Free modules

**Definition B.1 (*Generating set / R-basis / finitely generated/free R-module*)**

Let $M$ be an $R$-module and let $X \subseteq M$ be a subset. Then:

(a) $M$ is said to be **generated by** $X$ if every element $m \in M$ may be written as an $R$-linear combination $m = \sum_{x\in X} \lambda_x x$, i.e. where $\lambda_x \in R$ is almost everywhere 0. In this case we write $M = \langle X\rangle_R$ or $M = \sum_{x\in X} Rx$.

(b) $M$ is said to be **finitely generated** if it admits a finite set of generators.

(c) $X$ is an $R$-**basis** (or simply a **basis**) if $X$ generates $M$ and if every element of $M$ can be written *in a unique way* as an $R$-linear combination $\sum_{x\in X} \lambda_x X$ (i.e. with $\lambda_x \in R$ almost everywhere 0).

(d) $M$ is called **free** if it admits an $R$-basis $X$, and $|X|$ is called the $R$-**rank** of $M$.
**Notation:** In this case we write $M = \bigoplus_{x\in X} Rx \cong \bigoplus_{x\in X} R$.

**Remark B.2**

(a) **Warning**: If the ring $R$ is not commutative, then it is not true in general that two different bases of a free $R$-module have the same number of elements.

(b) Let $X$ be a generating set for $M$. Then, $X$ is a basis of $M$ if and only if $S$ is $R$-linearly independent.

(c) If $R$ is a field, then every $R$-module is free. ($R$-modules are $R$-vector spaces in this case!)

**Proposition B.3 (*Universal property of free modules*)**

Let $M$ be a free $R$-module with $R$-basis $X$. If $N$ is an $R$-module and $f : X \longrightarrow N$ is a map (of sets), then there exists a unique $R$-homomorphism $\widehat{f} : M \longrightarrow N$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "\circlearrowleft"] & N \\ M \arrow[ur, dashed, "\exists!\,\widehat{f}"'] & \end{tikzcd}$$

We say that $\widehat{f}$ is obtained by **extending $f$ by $R$-linearity**.

**Proof:** Given an $R$-linear combination $\sum_{x\in X}\lambda_x x \in M$, set $\widehat{f}(\sum_{x\in X}\lambda_x x) := \sum_{x\in X}\lambda_x f(x)$. The claim follows. ■

**Proposition B.4 (*Properties of free modules*)**

(a) Every $R$-module $M$ is isomorphic to a quotient of a free $R$-module.

(b) If $P$ is a free $R$-module, then $\mathrm{Hom}_R(P,-)$ is an exact functor.

## Projective modules

**Proposition-Definition B.5 (*Projective module*)**

Let $P$ be an $R$-module. Then the following are equivalent:

(a) The functor $\mathrm{Hom}_R(P,-)$ is exact.

(b) If $\psi \in \mathrm{Hom}_R(M,N)$ is a surjective morphism of $R$-modules, then the morphism of abelian groups $\psi_* : \mathrm{Hom}_R(P,M) \longrightarrow \mathrm{Hom}_R(P,N)$ is surjective.

(c) If $\pi : M \longrightarrow P$ is a surjective $R$-linear map, then $\pi$ splits, i.e., there exists $\sigma : P \longrightarrow M$ such that $\pi \circ \sigma = \mathrm{Id}_P$.

(d) $P$ is isomorphic to a direct summand of a free $R$-module.

If $P$ satisfies these equivalent conditions, then $P$ is called **projective**.

**Example B.6**

(a) If $R = \mathbb{Z}$, then every submodule of a free $\mathbb{Z}$-module is again free (main theorem on $\mathbb{Z}$-modules).

(b) Let $e$ be an idempotent in $R$, that is $e^2 = e$. Then, $R \cong Re \oplus R(1-e)$ and $Re$ is projective but not free if $e \neq 0, 1$.

(d) A direct sum of modules $\bigoplus_{i\in I} P_i$ is projective if and only if each $P_i$ is projective.

## C Direct products and direct sums

Let $\{M_i\}_{i\in I}$ be a family of $R$-modules. Then the abelian group $\prod_{i\in I} M_i$, that is the product of $\{M_i\}_{i\in I}$ seen as a family of abelian groups, becomes an $R$-module via the following external composition law:

$$\begin{aligned} R \times \prod_{i\in I} M_i &\longrightarrow \prod_{i\in I} M_i \\ \big(r, (m_i)_{i\in I}\big) &\longmapsto \big(r \cdot m_i\big)_{i\in I}. \end{aligned}$$

Furthermore, for each $j \in I$, we let $\pi_j : \prod_{i\in I} M_i \longrightarrow M_j, (m_i)_{i\in I} \mapsto m_j$ denotes the $j$-th projection from the product to the module $M_j$.

**Proposition C.1 (Universal property of the direct product)**

If $\{\varphi_i : L \longrightarrow M_i\}_{i\in I}$ is a family of $R$-homomorphisms, then there exists a unique $R$-homomorphism $\varphi : L \longrightarrow \prod_{i\in I} M_i$ such that $\pi_j \circ \varphi = \varphi_j$ for every $j \in I$.

$$\begin{tikzcd}
 & & L \arrow[d, dotted, "\varphi"] \arrow[ddl, bend right, "\varphi_k"'] \arrow[ddr, bend left, "\varphi_j"] & & \\
\cdots & \circlearrowleft & \prod_{i\in I} M_i \arrow[dl, "\pi_k"] \arrow[dr, "\pi_j"'] & \circlearrowleft & \cdots \\
 & M_k & & M_j &
\end{tikzcd}$$

Thus,

$$\begin{aligned} \mathrm{Hom}_R\Big(L, \prod_{i\in I} M_i\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}_R(L, M_i) \\ f &\longmapsto \big(\pi_i \circ f\big)_{i\in I} \end{aligned}$$

is an isomorphism of abelian groups.

Now let $\bigoplus_{i\in I} M_i$ be the subgroup of $\prod_{i\in I} M_i$ consisting of the elements $(m_i)_{i\in I}$ such that $m_i = 0$ almost everywhere (i.e. $m_i = 0$ exept for a finite subset of indices $i \in I$). This subgroup is called the **direct sum** of the family $\{M_i\}_{i\in I}$ and is in fact an $R$-submodule of the product. For each $j \in I$, we let $\eta_j : M_j \longrightarrow \bigoplus_{i\in I} M_i, m_j \mapsto$ denote the canonical injection of $M_j$ in the direct sum.

**Proposition C.2 (Universal property of the direct sum)**

If $\{f_i : M_i \longrightarrow L\}_{i\in I}$ is a family of $R$-homomorphisms, then there exists a unique $R$-homomorphism $\varphi : \bigoplus_{i\in I} M_i \longrightarrow L$ such that $f \circ \eta_j = f_j$ for every $j \in I$.

$$\begin{tikzcd}
 & L & \\
\cdots & \bigoplus_{i\in I} M_i \arrow[u, dotted, "f"] & \cdots \\
M_k \arrow[uur, bend left, "f_k"] \arrow[ur, "\eta_k"'] & & M_j \arrow[uul, bend right, "f_j"'] \arrow[ul, "\eta_j"]
\end{tikzcd}$$

Thus,

$$\begin{aligned} \mathrm{Hom}_R\Big(\bigoplus_{i\in I} M_i, L\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}_R(M_i, L) \\ f &\longmapsto \big(f \circ \eta_i\big)_{i\in I} \end{aligned}$$

is an isomorphism of abelian groups.

**Remark C.3**

It is clear that if $|I| < \infty$, then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$.

The direct sum as defined above is often called an *external* direct sum. This relates as follows with the usual notion of *internal* direct sum:

**Definition C.4 (*"Internal" direct sums*)**

Let $M$ be an $R$-module and $N_1, N_2$ be two $R$-submodules of $M$. We write $M = N_1 \oplus N_2$ if every $m \in M$ can be written in a unique way as $m = n_1 + n_2$, where $n_1 \in N_1$ and $n_2 \in N_2$.

In fact $M = N_1 \oplus N_2$ (internal direct sum) if and only if $M = N_1 + N_2$ and $N_1 \cap N_2 = \{0\}$.

**Proposition C.5**

If $N_1, N_2$ and $M$ are as above and $M = N_1 \oplus N_2$ then the homomorphism of $R$-modules

$$\begin{array}{rccl} \varphi\colon & M & \longrightarrow & N_1 \times N_2 = N_1 \oplus N_2 \quad \text{(external direct sum)} \\ & m = n_1 + n_2 & \mapsto & (n_1, n_2)\,, \end{array}$$

is an isomorphism of $R$-modules.

The above generalises to arbitrary internal direct sums $M = \bigoplus_{i\in I} N_i$.

# D Exact sequences

Exact sequences constitute a very useful tool for the study of modules. Often we obtain valuable information about modules by *plugging them* in short exact sequences, where the other terms are known.

**Definition D.1 (*Exact sequence*)**

A sequence $L \xrightarrow{\varphi} M \xrightarrow{\psi} N$ of $R$-modules and $R$-linear maps is called **exact (at** $M$**)** if $\operatorname{Im}\varphi = \ker\psi$.

**Remark D.2 (*Injectivity/surjectivity/short exact sequences*)**

(a) $L \xrightarrow{\varphi} M$ is injective $\iff 0 \longrightarrow L \xrightarrow{\varphi} M$ is exact at $L$.

(b) $M \xrightarrow{\psi} N$ is surjective $\iff M \xrightarrow{\psi} N \longrightarrow 0$ is exact at $N$.

(c) $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is exact (i.e. at $L$, $M$ and $N$) if and only if $\varphi$ is injective, $\psi$ is surjective and $\psi$ induces an $R$-isomorphism $\overline{\psi} : M/\operatorname{Im}\varphi \longrightarrow N, m + \operatorname{Im}\varphi \mapsto \psi(m)$.
Such a sequence is called a **short exact sequence (s.e.s.** for short).

(d) If $\varphi \in \operatorname{Hom}_R(L, M)$ is an injective morphism, then there is a s.e.s.

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\pi} \operatorname{coker}(\varphi) \longrightarrow 0$$

where $\pi$ is the canonical projection.

(e) If $\psi \in \operatorname{Hom}_R(M, N)$ is a surjective morphism, then there is a s.e.s.

$$0 \longrightarrow \ker(\psi) \xrightarrow{i} M \xrightarrow{\psi} N \longrightarrow 0\,,$$

where $i$ is the canonical injection.

**Proposition D.3**

Let $Q$ be an $R$-module. Then the following holds:

(a) $\operatorname{Hom}_R(Q, -) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact covariant functor. In other words, if $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is a s.e.s of $R$-modules, then the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(Q, L) \xrightarrow{\varphi_*} \operatorname{Hom}_R(Q, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(Q, N)$$

is an exact sequence of abelian groups. Here $\varphi_* := \operatorname{Hom}_R(Q, \varphi)$, that is $\varphi_*(\alpha) = \varphi \circ \alpha$ for every $\alpha \in \operatorname{Hom}_R(Q, L)$ and similarly for $\psi_*$.

(b) $\operatorname{Hom}_R(-, Q) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact contravariant functor. In other words, if $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is a s.e.s of $R$-modules, then the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, Q) \xrightarrow{\psi^*} \operatorname{Hom}_R(M, Q) \xrightarrow{\varphi^*} \operatorname{Hom}_R(L, Q)$$

is an exact sequence of abelian groups. Here $\varphi^* := \operatorname{Hom}_R(\varphi, Q)$, that is $\varphi^*(\alpha) = \alpha \circ \varphi$ for every $\alpha \in \operatorname{Hom}_R(M, Q)$ and similarly for $\psi^*$.

Exercise: verify that $\operatorname{Hom}_R(Q, -)$ and $\operatorname{Hom}_R(-, Q)$ are functors.
Notice that $\operatorname{Hom}_R(Q, -)$ and $\operatorname{Hom}_R(-, Q)$ are not *right* exact in general. Exercise: find counter-examples!

**Lemma-Definition D.4 (*Split short exact sequence*)**

A s.e.s. $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ of $R$-modules is called **split** if it satisfies one of the following equivalent conditions:

(a) $\psi$ admits an $R$-linear section, i.e. if $\exists\, \sigma \in \mathrm{Hom}_R(N, M)$ such that $\psi \circ \sigma = \mathrm{Id}_N$;

(b) $\varphi$ admits an $R$-linear retraction, i.e. if $\exists\, \rho \in \mathrm{Hom}_R(M, L)$ such that $\rho \circ \varphi = \mathrm{Id}_L$;

(c) $\exists$ an $R$-isomorphism $\alpha : M \longrightarrow L \oplus N$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id}_L"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r, "\psi"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & N \arrow[r] \arrow[d, "\mathrm{Id}_N"] & 0 \\ 0 \arrow[r] & L \arrow[r, "i"] & L \oplus N \arrow[r, "p"] & N \arrow[r] & 0\,, \end{tikzcd}$$

where $i$, resp. $p$, are the canonical inclusion, resp. projection.

**Remark D.5**

If the sequence splits and $\sigma$ is a section, then $M = \varphi(L) \oplus \sigma(N)$. If the sequence splits and $\rho$ is a retraction, then $M = \varphi(L) \oplus \ker(\rho)$.

**Example D.6**

The s.e.s. of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

defined by $\varphi([1]) = ([1], [0])$ and where $\pi$ is the canonical projection onto the cokernel of $\varphi$ is split but the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

defined by $\varphi([1]) = ([2])$ and $\pi$ is the canonical projection onto the cokernel of $\varphi$ is not split. Exercise: justify this fact using a straightforward argument.

# E Tensor products

**Definition E.1 (*Tensor product of R-modules*)**

Let $M$ be a right $R$-module and let $N$ be a left $R$-module. Let $F$ be the free abelian group (= free $\mathbb{Z}$-module) with basis $M \times N$. Let $G$ be the subgroup of $F$ generated by all the elements

$$\begin{aligned} &(m_1 + m_2, n) - (m_1, n) - (m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N, \\ &(m, n_1 + n_2) - (m, n_1) - (m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and} \\ &(mr, n) - (m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R. \end{aligned}$$

The **tensor product of $M$ and $N$ (balanced over $R$)**, is the abelian group $M \otimes_R N := F/G$. The class of $(m,n) \in F$ in $M \otimes_R N$ is denoted by $m \otimes n$.

**Remark E.2**

(a) $M \otimes_R N = \langle m \otimes n \mid m \in M, n \in N \rangle_{\mathbb{Z}}$.

(b) In $M \otimes_R N$, we have the relations

$$\begin{aligned} (m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n, \quad \forall m_1, m_2 \in M, \forall n \in N, \\ m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2, \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and} \\ mr \otimes n &= m \otimes rn, \quad \forall m \in M, \forall n \in N, \forall r \in R. \end{aligned}$$

In particular, $m \otimes 0 = 0 = 0 \otimes n \ \forall\ m \in M, \forall\ n \in N$ and $(-m) \otimes n = -(m \otimes n) = m \otimes (-n)$ $\forall\ m \in M, \forall\ n \in N$.

**Definition E.3 (*$R$-balanced map*)**

Let $M$ and $N$ be as above and let $A$ be an abelian group. A map $f : M \times N \longrightarrow A$ is called **$R$-balanced** if

$$\begin{aligned} f(m_1 + m_2, n) &= f(m_1, n) + f(m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N, \\ f(m, n_1 + n_2) &= f(m, n_1) + f(m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N, \\ f(mr, n) &= f(m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R. \end{aligned}$$

**Remark E.4**

The canonical map $t : M \times N \longrightarrow M \otimes_R N, (m,n) \mapsto m \otimes n$ is $R$-balanced.

**Proposition E.5 (*Universal property of the tensor product*)**

Let $M$ be a right $R$-module and let $N$ be a left $R$-module. For every abelian group $A$ and every $R$-balanced map $f : M \times N \longrightarrow A$ there exists a unique $\mathbb{Z}$-linear map $\overline{f} : M \otimes_R N \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & A \\ M \otimes_R N \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

**Proof:** Let $\iota : M \times N \longrightarrow F$ denote the canonical inclusion, and let $\pi : F \longrightarrow F/G$ denote the canonical projection. By the universal property of the free $\mathbb{Z}$-module, there exists a unique $\mathbb{Z}$-linear map $\tilde{f} : F \longrightarrow A$ such that $\tilde{f} \circ \iota = f$. Since $f$ is $R$-balanced, we have that $G \subseteq \ker(\tilde{f})$. Therefore, the universal property of the quotient yields the existence of a unique homomorphism of abelian groups $\overline{f} : F/G \longrightarrow A$ such that $\overline{f} \circ \pi = \tilde{f}$:

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\iota"] \arrow[dd, bend right, "t"'] & A \\ F \arrow[ur, dashed, "\tilde{f}"] \arrow[d, "\pi"] & \\ M \otimes_R N \cong F/G \arrow[uur, dashed, "\overline{f}"'] & \end{tikzcd}$$

Clearly $t = \pi \circ \iota$, and hence $\overline{f} \circ t = \overline{f} \circ \pi \circ \iota = \tilde{f} \circ \iota = f$. ■

**Remark E.6**

Let $M$ and $N$ be as in Definition E.1.

(a) Let $\{M_i\}_{i\in I}$ be a collection of right $R$-modules, $M$ be a right $R$-module, $N$ be a left $R$-module and $\{N_j\}_{i\in J}$ be a collection of left $R$-modules. Then, we have

$$\bigoplus_{i\in I} M_i \otimes_R N \cong \bigoplus_{i\in I}(M_i \otimes_R N)$$
$$M \otimes_R \bigoplus_{j\in J} N_j \cong \bigoplus_{j\in J}(M \otimes_R N_j).$$

(This is easily proved using both the universal property of the direct sum and of the tensor product.)

(b) There are natural isomorphisms of abelian groups given by $R \otimes_R N \cong N$ via $r \otimes n \mapsto rn$, and $M \otimes_R R \cong M$ via $m \otimes r \mapsto mr$.

(c) It follows from (b), that if $P$ is a free left $R$-module with $R$-basis $X$, then $N \otimes_R P \cong \bigoplus_{x\in X} N$, and if $P$ is a free right $R$-module with $R$-basis $X$, then $P \otimes_R M \cong \bigoplus_{x\in X} M$.

(d) Let $Q$ be a third ring. Then we obtain module structures on the tensor product as follows:

(i) If $M$ is a $(Q, R)$-bimodule and $N$ a left $R$-module, then $M \otimes_R N$ can be endowed with the structure of a left $Q$-module via

$$q \cdot (m \otimes n) = q \cdot m \otimes n \quad \forall q \in Q, \forall m \in M, \forall n \in N.$$

(ii) If $M$ is a right $R$-module and $N$ an $(R, S)$-bimodule, then $M \otimes_R N$ can be endowed with the structure of a right $S$-module via

$$(m \otimes n) \cdot s = qm \otimes n \cdot s \quad \forall s \in S, \forall m \in M, \forall n \in N.$$

(iii) If $M$ is a $(Q, R)$-bimodule and $N$ an $(R, S)$-bimodule. Then $M \otimes_R N$ can be endowed with the structure of a $(Q, S)$-bimodule via the external composition laws defined in (i) and (ii).

(e) Assume $R$ is commutative. Then any $R$-module can be viewed as an $(R, R)$-bimodule. Then, in particular, $M \otimes_R N$ becomes an $R$-module (both on the left and on the right).

(f) For instance, it follows from (e) that if $K$ is a field and $M$ and $N$ are $K$-vector spaces with $K$-bases $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ resp., then $M \otimes_K N$ is a $K$-vector space with a $K$-basis given by $\{x_i \otimes y_j\}_{(i,j)\in I\times J}$.

(g) **Tensor product of morphisms:** Let $f : M \longrightarrow M'$ be a morphism of right $R$-modules and $g : N \longrightarrow N'$ be a morphism of left $R$-modules. Then, by the universal property of the tensor product, there exists a unique $\mathbb{Z}$-linear map $f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

**Exercise E.7**

(a) Assume $R$ is a commutative ring and $I$ is an ideal of $R$. Let $M$ be a left $R$-module. Prove that there is an isomorphism of left $R$-modules $R/I \otimes_R M \cong M/IM$.

(b) Let $m, n$ be coprime positive integers. Compute $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(c) Let $K$ be a field and let $U, V$ be finite-dimensional $K$-vector spaces. Prove that there is a natural isomorphism of $K$-vector spaces:

$$\mathrm{Hom}_K(U, V) \cong U^* \otimes_K V\,.$$

**Proposition E.8 (*Right exactness of the tensor product*)**

(a) Let $N$ be a left $R$-module. Then $- \otimes_R N : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$ is a right exact covariant functor.

(b) Let $M$ be a right $R$-module. Then $M \otimes_R - : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a right exact covariant functor.

**Remark E.9**

The functors $- \otimes_R N$ and $M \otimes_R -$ are not left exact in general.

# F Algebras

In this lecture we aim at studying modules over specific rings, which are in particular *algebras*.

**Definition F.1 (*Algebra*)**

Let $R$ be a commutative ring.

(a) An **$R$-algebra** is an ordered quadruple $(A, +, \cdot, *)$ such that the following axioms hold:

**(A1)** $(A, +, \cdot)$ is a ring;

**(A2)** $(A, +, *)$ is a left $R$-module; and

**(A3)** $r * (a \cdot b) = (r * a) \cdot b = a \cdot (r * b)\ \forall\ a, b \in A,\ \forall\ r \in R$.

(b) A map $f : A \to B$ between two $R$-algebras is called an **algebra homomorphism** iff:

(i) $f$ is a homomorphism of $R$-modules;

(ii) $f$ is a ring homomorphism.

**Example F.2 (*Algebras*)**

(a) The ring $R$ itself is an $R$-algebra.
[The internal composition law "$\cdot$" and the external composition law "$*$" coincide in this case.]

(b) For each $n \in \mathbb{Z}_{\geqslant 1}$ the set $M_n(R)$ of $n \times n$-matrices with coefficients in $R$ is an $R$-algebra for its usual $R$-module and ring structures.

[Note: in particular $R$-algebras need not be commutative rings in general!]

(c) Let $K$ be a field. Then for each $n \in \mathbb{Z}_{\geqslant 1}$ the polynom ring $K[X_1, \dots, X_n]$ is a $K$-algebra for its usual $K$-vector space and ring structure.

(d) $\mathbb{R}$ and $\mathbb{C}$ are $\mathbb{Q}$-algebras, $\mathbb{C}$ is an $\mathbb{R}$-algebra, ...

(e) Rings are $\mathbb{Z}$-algebras.
Exercise: Check it!

**Example F.3 (*Modules over algebras*)**

(a) $A = M_n(R) \Rightarrow R^n$ is an $A$-module for the external composition law given by left matrix multiplication $A \times R^n \longrightarrow R^n, (B, x) \mapsto Bx$.

(b) If $K$ is a field and $V$ a $K$-vector space, then $V$ becomes an $A$-algebra for $A := \mathrm{End}_K(V)$ together with the external composition law

$$A \times V \longrightarrow V, (\varphi, v) \mapsto \varphi(v)\,.$$

Exercise: Check it!

(c) An arbitrary $A$-module $M$ can be seen as an $R$-module via a change of the base ring since $R \longrightarrow A, r \mapsto r * 1_A$ is a homomorphism of rings by the algebra axioms.

**Exercise F.4**

Let $R$ be a commutative ring.

(a) Let $M, N$ be $R$-modules. Prove that:

(1) $\mathrm{End}_R(M)$, endowed with the pointwise addition of maps and the usual composition of maps, is a ring. (Note that the commutativity of $R$ is not necessary!)

(2) The abelian group $\mathrm{Hom}_R(M, N)$ is a left $R$-module for the external composition law defined by

$$(rf)(m) := f(rm) = rf(m) \qquad \forall\, r \in R,\ \forall f \in \mathrm{Hom}_R(M, N),\ \forall m \in M\,.$$

(b) Let now $A$ be an $R$-algebra and $M$ be an $A$-module. Prove that $\mathrm{End}_R(M)$ and $\mathrm{End}_A(M)$ are $R$-algebras.

# Appendix 2: The Language of Category Theory

This appendix gives a short introduction to some of the basic notions of category theory used in this lecture.

## G Categories

**Definition G.1 (*Category*)**

A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\,\mathcal{C}$ of **objects**,
- a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of **morphisms** for every ordered pair $(A, B)$ of objects, and
- a **composition function**

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,C) \\ (f,g) & \mapsto & g \circ f \end{array}$$

  for each ordered triple $(A, B, C)$ of objects,

satisfying the following axioms:

**(C1)** **Unit axiom**: for each object $A \in \mathrm{Ob}\,\mathcal{C}$, there exists an **identity morphism** $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all $B \in \mathrm{Ob}\,\mathcal{C}$,

$$f \circ 1_A = f = 1_B \circ f\,.$$

**(C2)** **Associativity axiom**: for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ with $A, B, C, D \in \mathrm{Ob}\,\mathcal{C}$,

$$h \circ (g \circ f) = (h \circ g) \circ f\,.$$

Let us start with some remarks and examples to enlighten this definition:

**Remark G.2**

(a) $\mathrm{Ob}\,\mathcal{C}$ need not be a set!

(b) The only requirement on $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is that it be a set, and it is allowed to be empty.

(c) It is common to write $f : A \longrightarrow B$ or $A \xrightarrow{f} B$ instead of $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$, and to talk about *arrows* instead of *morphisms*. It is also common to write "$A \in \mathcal{C}$" instead of "$A \in \operatorname{Ob} \mathcal{C}$".

(d) The identity morphism $1_A \in \operatorname{Hom}_{\mathcal{C}}(A, A)$ is uniquely determined: indeed, if $f_A \in \operatorname{Hom}_{\mathcal{C}}(A, A)$ were a second identity morphisms, then we would have $f_A = f_A \circ 1_A = 1_A$.

**Example G.3**

(a) $\mathcal{C} = \mathbf{1}$ : category with one object and one morphism (the identity morphism):

$$\begin{tikzcd} \bullet \arrow[loop right, "1_\bullet"] \end{tikzcd}$$

(b) $\mathcal{C} = \mathbf{2}$ : category with two objects and three morphism, where two of them are identity morphisms and the third one goes from one object to the other:

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r] & B \arrow[loop right, "1_B"] \end{tikzcd}$$

(c) A group $G$ can be seen as a category $\mathcal{C}(G)$ with one object: $\operatorname{Ob} \mathcal{C}(G) = \{\bullet\}$, $\operatorname{Hom}_{\mathcal{C}(G)}(\bullet, \bullet) = G$ (notice that this is a set) and composition is given by multiplication in the group.

(d) The $n \times m$-matrices with entries in a field $k$ for $n, m$ ranging over the positive integers form a category $\mathbf{Mat}_k$: $\operatorname{Ob} \mathbf{Mat}_k = \mathbb{Z}_{>0}$, morphisms $n \longrightarrow m$ from $n$ to $m$ are the $m \times n$-matrices, and compositions are given by the ordinary matrix multiplication.

**Example G.4 (*Categories and algebraic structures*)**

(a) $\mathcal{C} = \mathbf{Set}$, the *category of sets*: objects are sets, morphisms are maps of sets, and composition is the usual composition of functions.

(b) $\mathcal{C} = \mathbf{Vec}_k$, the *category of vector spaces over the field $k$*: objects are $k$-vector spaces, morphisms are $k$-linear maps, and composition is the usual composition of functions.

(c) $\mathcal{C} = \mathbf{Top}$, the *category of topological spaces*: objects are topological spaces, morphisms are continous maps, and composition is the usual composition of functions.

(d) $\mathcal{C} = \mathbf{Grp}$, the *category of groups*: objects are groups, morphisms are homomorphisms of groups, and composition is the usual composition of functions.

(e) $\mathcal{C} = \mathbf{Ab}$, the *category of abelian groups*: objects are abelian groups, morphisms are homomorphisms of groups, and composition is the usual composition of functions.

(f) $\mathcal{C} = \mathbf{Rng}$, the *category of rings*: objects are rings, morphisms are homomorphisms of rings, and composition is the usual composition of functions.

(g) $\mathcal{C} = {}_R\mathbf{Mod}$, the *category of left R-modules*: objects are *left* modules over the ring $R$, morphisms are $R$-homomorphisms, and composition is the usual composition of functions.

(g′) $\mathcal{C} = \mathbf{Mod}_R$, the *category of left R-modules*: objects are *right* modules over the ring $R$, morphisms are $R$-homomorphisms, and composition is the usual composition of functions.

(g″) $\mathcal{C} = {}_R\mathbf{Mod}_S$, the *category of* $(R, S)$*-bimodules*: objects are $(R, S)$-bimodules over the rings $R$ and $S$, morphisms are $(R, S)$-homomorphisms, and composition is the usual composition of functions.

(h) Examples of your own ...

**Definition G.5 (*Monomorphism/epimorphism*)**

Let $\mathcal{C}$ be a category and let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ be a morphism. Then $f$ is called

(a) a **monomorphism** iff for all morphisms $g_1, g_2 : C \longrightarrow A$,

$$f \circ g_1 = f \circ g_2 \Longrightarrow g_1 = g_2 \,.$$

(b) an **epimorphism** iff for all morphisms $g_1, g_2 : B \longrightarrow C$,

$$g_1 \circ f = g_2 \circ f \Longrightarrow g_1 = g_2 \,.$$

**Remark G.6**

In categories, where morphisms are set-theoretic maps, then injective morphisms are monomorphisms, and surjective morphisms are epimorphisms.
In module categories (${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$, ${}_R\mathbf{Mod}_S$, ...), the converse holds as well, but:

**Warning**: It is not true in general, that all monomorphisms must be injective, and all epimorphisms must be surjective.

For example in **Rng**, the canonical injection $\iota : \mathbb{Z} \longrightarrow \mathbb{Q}$ is an epimorphism. Indeed, if $C$ is a ring and $g_1, g_2 \in \mathrm{Hom}_{\mathbf{Rng}}(\mathbb{Q}, C)$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\iota"] & \mathbb{Q} \arrow[r, shift left, "g_2"] \arrow[r, shift right, "g_1"'] & C \end{tikzcd}$$

are such that $g_1 \circ \iota = g_2 \circ \iota$, then we must have $g_1 = g_2$ by the universal property of the field of fractions. However, $\iota$ is clearly not surjective.

# H Functors

**Definition H.1 (*Covariant functor*)**

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a collection of maps:

- $F : \operatorname{Ob}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{D}, X \mapsto F(X)$, and
- $F_{A,B} : \operatorname{Hom}_{\mathcal{C}}(A, B) \mapsto \operatorname{Hom}_{\mathcal{D}}(F(A), F(B))$,

satisfying:

(a) If $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$; and

(b) $F(1_A) = 1_{F(A)}$ for every $A \in \operatorname{Ob}\mathcal{C}$.

**Definition H.2 (*Contravariant functor*)**

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **contravariant functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a collection of maps:

- $F : \operatorname{Ob}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{D}, X \mapsto F(X)$, and
- $F_{A,B} : \operatorname{Hom}_{\mathcal{C}}(A, B) \mapsto \operatorname{Hom}_{\mathcal{D}}(F(B), F(A))$,

satisfying:

(a) If $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathcal{C}$, then $F(g \circ f) = F(f) \circ F(g)$; and

(b) $F(1_A) = 1_{F(A)}$ for every $A \in \operatorname{Ob}\mathcal{C}$.

**Remark H.3**

Often in the literature functors are defined only on objects of categories. When no confusion is to be made and the action of functors on the morphism sets are implicitely obvious, we will also adopt this convention.

**Example H.4**

Let $Q \in \operatorname{Ob}({}_R\mathbf{Mod})$. Then

$$\begin{array}{rccc} \operatorname{Hom}_R(Q, -): & {}_R\mathbf{Mod} & \longrightarrow & \mathbf{Ab} \\ & M & \mapsto & \operatorname{Hom}_R(Q, M)\,, \end{array}$$

is a covariant functor, and

$$\begin{array}{rccc} \operatorname{Hom}_R(-, Q): & {}_R\mathbf{Mod} & \longrightarrow & \mathbf{Ab} \\ & M & \mapsto & \operatorname{Hom}_R(M, Q)\,, \end{array}$$

is a contravariant functor.

**Exact Functors.**

We are now interested in the relations between functors and exact sequences in categories where it makes sense to define exact sequences, that is categories that behave essentially like module categories such as ${}_R\mathbf{Mod}$. These are the so-called **abelian categories**. It is not the aim, to go into these details, but roughly speaking abelian categories are categories satisfying the following properties:

- they have a zero object (in ${}_R\mathbf{Mod}$: the zero module)
- they have products and coproducts (in ${}_R\mathbf{Mod}$: products and direct sums)
- they have kernels and cokernels (in ${}_R\mathbf{Mod}$: the usual kernels and cokernels of $R$-linear maps)
- monomorphisms are kernels and epimorphisms are cokernels (in ${}_R\mathbf{Mod}$: satisfied)

**Definition H.5 (*Pre-additive categories/additive functors*)**

(a) A category $\mathcal{C}$ in which all sets of morphisms are abelian groups is called **pre-additive**.

(b) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between pre-additive categories is called **additive** iff the maps $F_{A,B}$ are homomorphisms of groups for all $A, B \in \operatorname{Ob}\mathcal{C}$.

**Definition H.6 (*Left exact/right exact/exact functors*)**

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant (resp. contravariant) additive functor between two abelian categories, and let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a s.e.s. of objects and morphisms in $\mathcal{C}$. Then $F$ is called:

(a) **left exact** if $0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ (resp. $0 \longrightarrow F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$)) is an exact sequence.

(b) **right exact** if $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$ (resp. $F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)) \longrightarrow 0$) is an exact sequence.

(c) **exact** if $0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$ (resp. $0 \longrightarrow F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)) \longrightarrow 0$) is a short exact sequence.

**Example H.7**

The functors $\operatorname{Hom}_R(Q, -)$ and $\operatorname{Hom}_R(-, Q)$ of Example H.4 are both left exact functors. Moreover $\operatorname{Hom}_R(Q, -)$ is exact if and only if $Q$ is projective, and $\operatorname{Hom}_R(-, Q)$ is exact if and only if $Q$ is injective.