JUN.-PROF. DR. CAROLINE LASSUEUR | **FB MATHEMATIK**
BERNHARD BÖHMLER
Due date: – | WS 2020/21

Throughout, G denotes a finite group and $(F, \mathcal{O}, k)$ is a p-modular system such that F contains an $\exp(G)$-th root of unity. Furthermore, all modules considered are assumed to be *left* modules and finitely generated.

EXERCISE 1.
Let U, V, W be non-zero kG-modules. Prove that the following assertions.

(a) If $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is a s.e.s. of kG-modules, then

$$\varphi_V = \varphi_U + \varphi_W\,.$$

(b) If the composition factors of U are $S_1, \ldots, S_m$ ($m \in \mathbb{Z}_{\geq 1}$) with multiplicities $n_1, \ldots, n_m$ respectively, then

$$\varphi_U = n_1\varphi_{S_1} + \ldots + n_m\varphi_{S_m}\,.$$

In particular, if two kG-modules have isomorphic composition factors, counting multiplicities, then they have the same Brauer character.

(c) $\varphi_{U\oplus V} = \varphi_U + \varphi_V$ and $\varphi_{U\otimes_k V} = \varphi_U \cdot \varphi_V$.

EXERCISE 2.
Prove that two kG-modules afford the same Brauer character if and only if they have isomorphic composition factors (including multiplicities).

EXERCISE 3.
Let H be a p'-subgroup of a finite group G. Prove that the character Φ_k is a constituent of the trivial F-character of H induced to G.

EXERCISE 4.
Let $\varphi, \lambda \in \mathrm{IBr}_p(G)$ and assume that λ is linear. Prove that $\lambda\varphi \in \mathrm{IBr}_p(G)$ and $\lambda\Phi_\varphi = \Phi_{\lambda\varphi}$.

EXERCISE 5.
Let G be a finite group and let ρ_{reg} denote the regular F-character of G. Prove that:

$$\rho_{\mathrm{reg}} = \sum_{\varphi\in\mathrm{IBr}_p(G)} \varphi(1)\Phi_\varphi \quad \text{and} \quad (\rho_{\mathrm{reg}})|_{G_{p'}} = \sum_{\varphi\in\mathrm{IBr}_p(G)} \Phi_\varphi(1)\varphi\,.$$

Exercise 6.
Prove that:

(a) the inverse of the Cartan matrix of kG is $C^{-1} = (\langle \varphi, \psi \rangle_{p'})_{\varphi,\psi \in \mathrm{IBr}_p(G)}$; and

(b) $|G|_p \mid \Phi_\varphi(1)$ for every $\varphi \in \mathrm{IBr}_p(G)$.

Exercise 7.
Let U be a kG-module and let P be a PIM of kG. Prove that

$$\dim_k \mathrm{Hom}_{kG}(P, U) = \frac{1}{|G|} \sum_{g \in G_{p'}} \varphi_P(g^{-1})\varphi_U(g)$$

Exercise 8.
Let $G := \mathfrak{A}_5$, the alternating group on 5 letters. Calculate the Brauer character table, the Cartan matrix and the decomposition matrix of G for $p = 3$.
[Hints. (1.) Use the ordinary character table of $\mathfrak{A}_5$ and reduction modulo p. (2.) A simple group does not have any irreducible Brauer character of degree 2.]

Exercise 9.
Deduce from Remark 35.12 of the Lecture Notes that column orthogonality relations for the Brauer characters take the form $\overline{\Pi}^{tr}\Phi = B$, i.e. given $g, h \in G_{p'}$ we have

$$\sum_{\phi \in \mathrm{IBr}_p(G)} \phi(g)\Phi_\varphi(h^{-1}) = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are } G\text{-conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$