Jun.-Prof. Dr. Caroline Lassueur **FB Mathematik**
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Throughout, G denotes a finite group. Furthermore, all modules considered are assumed to be *left* modules and finitely generated.

A. Exercises for the tutorial.

Exercise 1.
Let $(F, \mathcal{O}, k)$ be a p-modular system and write $\mathfrak{p} := J(\mathcal{O})$. Let L be an $\mathcal{O}G$-module. Verify that

- setting $L^F := F \otimes_{\mathcal{O}} L$ defines an FG-module, and
- reduction modulo $\mathfrak{p}$ of L, that is $\overline{L} := L/\mathfrak{p}L \cong k \otimes_{\mathcal{O}} L$ defines a kG-module.

Exercise 2.
Let $\mathcal{O}$ be a complete discrete valuation ring and let $F := \mathrm{Frac}(\mathcal{O})$ be the fraction field of $\mathcal{O}$. Let V be a finitely generated FG-module and let $\{v_1, \ldots, v_n\}$ be an F-basis of V. Prove that $L := \mathcal{O}Gv_1 + \cdots + \mathcal{O}Gv_n \subseteq V$ is an $\mathcal{O}$-form of V.

Exercise 3.
Let $\mathcal{O}$ be a commutative ring. Let A be a finitely-generated $\mathcal{O}$-algebra of finite $\mathcal{O}$-rank and let $e \in A$ be an idempotent element. Let V be an A-module. Prove that

$$\mathrm{Hom}_A(Ae, V) \cong eV$$

as $\mathrm{End}_A(V)$-modules.

B. Exercises to hand in.

Exercise 4.
Let $\mathcal{O}$ be a local commutative ring with unique maximal ideal $\mathfrak{p} := J(\mathcal{O})$ and residue field $k := \mathcal{O}/J(\mathcal{O})$.

(a) Let M, N be finitely generated free $\mathcal{O}$-modules.

  (i) Let $f : M \longrightarrow N$ is an $\mathcal{O}$-linear map and $\overline{f} : \overline{M} \longrightarrow \overline{N}$ its reduction modulo $\mathfrak{p}$. Prove that if $\overline{f}$ is surjective (resp. an isomorphism), then f is surjective (resp. an isomorphism).

  (ii) Prove that if elements $x_1, \dots, x_n \in M$ ($n \in \mathbb{Z}_{\geq 1}$) are such that their images $\overline{x}_1, \dots \overline{x}_2 \in \overline{M}$ form a k-basis of $\overline{M}$, then $\{x_1, \dots, x_n\}$ is an $\mathcal{O}$-basis of M.
  In particular, $\dim_k(\overline{M}) = \mathrm{rk}_{\mathcal{O}}(M)$.

Deduce that any direct summand of a finitely generated free $\mathcal{O}$-module is free.

(b) Prove that if M is a finitely generated $\mathcal{O}$-module, then the following conditions are equivalent:

  (i) M is projective;

  (ii) M is free.

[Hint: Use Nakayama's Lemma.]

Exercise 5.
Let $\mathcal{O}$ be a complete discrete valuation ring. Let A and B be a finitely generated $\mathcal{O}$-algebras of finite $\mathcal{O}$-rank and let $f : A \twoheadrightarrow B$ be a surjective $\mathcal{O}$-algebra homomorphism. Prove that:

(a) f maps $J(A)$ onto $J(B)$; and

(b) f maps $A^{\times}$ onto $B^{\times}$.

Exercise 6.
Let $\mathcal{O}$ be a complete discrete valuation ring and write $\mathfrak{p} := J(\mathcal{O})$. Let A be a finitely generated $\mathcal{O}$-algebra of finite $\mathcal{O}$-rank. Set $\overline{A} := A/\mathfrak{p}A$ and for $a \in A$ write $\overline{a} := a + \mathfrak{p}A$. Prove that:

(a) For every idempotent $x \in \overline{A}$, there exists an idempotent $e \in A$ such that $\overline{e} = x$.

(b) $A^{\times} = \{a \in A \mid \overline{a} \in \overline{A}^{\times}\}$.

(c) If $e_1, e_2 \in A$ are idempotents such that $\overline{e}_1 = \overline{e}_2$ then there is a unit $u \in A^{\times}$ such that $e_1 = ue_2u^{-1}$.

(d) The quotient morphism $A \to \overline{A}$ induces a bijection between the central idempotents of A and the central idempotents of $\overline{A}$.