REPRESENTATION THEORY — EXERCISE SHEET 5 | **TU KAISERSLAUTERN**

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Throughout, K denotes a commutative ring and G a finite group. Furthermore, all KG-modules considered are assumed to be *left* modules and free of finite rank over K.

A. Exercises for the tutorial.

EXERCISE 1.
Let $H \leq J \leq G$. Let U be a KG-module and let V be a KJ-module. Prove the following statements.

(a) If U is H-projective then U is J-projective.

(b) If U is a direct summand of $V\uparrow_J^G$ and V is H-projective, then U is H-projective.

(c) For any $g \in G$, U is H-projective if and only if ${}^gU$ is ${}^gH$-projective.

(d) If U is H-projective and W is any KG-module, then $U \otimes_K W$ is H-projective.
[Hint: use part (f) of Proposition 25.4.]

EXERCISE 2.
Assume K is a field of characteristc $p > 0$ and let A, B, C, U, V be KG-modules. Prove that:

(a) Any direct summand of a V-projective KG-module is V-projective;

(b) If $U \in \operatorname{Proj}(V)$, then $\operatorname{Proj}(U) \subseteq \operatorname{Proj}(V)$;

(c) If $p \nmid \dim_K(V)$ then any KG-module is V-projective;

(d) $\operatorname{Proj}(V) = \operatorname{Proj}(V^*)$;

(e) $\operatorname{Proj}(U \oplus V) = \operatorname{Proj}(U) \oplus \operatorname{Proj}(V)$;

(f) $\operatorname{Proj}(U) \cap \operatorname{Proj}(V) = \operatorname{Proj}(U \otimes_K V)$;

(g) $\operatorname{Proj}(\bigoplus_{j=1}^n V) = \operatorname{Proj}(V) = \operatorname{Proj}(\bigotimes_{j=1}^m V)\ \forall\, m, n \in \mathbb{Z}_{>0}$;

(h) $C \cong A \oplus B$ is V-projective if and only if both A and B are V-projective;

(i) $\operatorname{Proj}(V) = \operatorname{Proj}(V^* \otimes_K V)$.

Hint: Use Lemma 13.8 and Exercise 4(c) on Sheet 3. Proceed in the given order.

B. Exercises to hand in.

Assume now that K is a field of positive characteristic p.

Exercise 3.
Let M be a KG-module with dimension coprime to p. Prove that the vertices of M are the Sylow p-subgroups G.

Exercise 4.
Let $Q \leq G$ be a p-subgroup and let $L \leq G$. Prove the following assertions.

(a) If U is an indecomposable KG-module with vertex Q and $L \geq Q$, then there exists an indecomposable direct summand of $U\downarrow_L^G$ with vertex Q.

(b) If $L \geq N_G(Q)$, then the following assertions hold.

  (i) If V is an indecomposable KL-module with vertex Q and U is a direct summand of $V\uparrow_L^G$ such that $V \mid U\downarrow_L^G$, then Q is also a vertex of U.

  (ii) If V is an indecomposable KL-module which is Q-projective and there exists an indecomposable direct summand U of $V\uparrow_L^G$ with vertex Q, then Q is also a vertex of V.

Exercise 5.

(a) Verify that modules corresponding to each other via the Green correspondence have a source in common.

(b) Compute the Green correspondent of the trivial module.