Jun.-Prof. Dr. Caroline Lassueur | **FB Mathematik**
Bernhard Böhmler

Due date: **Wednesday, 9th of December 2020, 10 a.m.** | WS 2020/21

Throughout, K denotes a **field** and G a finite group. Furthermore, all KG-modules considered are assumed to be *left* modules and finite-dimensional over K.
Recall that $M \mid N$ means that the KG-module M is (isomorphic to) a direct summand of the KG-module N.

A. Exercises for the tutorial.

Exercise 1 (Proof of the Converse of Maschke's Theorem for K splitting field for G.).
Assume K is a splitting field for G of positive characteristic p with $p \mid |G|$. Set $T := \langle \sum_{g \in G} g \rangle_K$.

(a) Prove that there is a series of KG-submodules given by $KG^\circ \supsetneq I(KG) \supseteq T \supsetneq 0$.

(b) Deduce that the regular module KG° has at least two composition factors isomorphic to the trivial module K.

(c) Deduce that KG is not a semisimple K-algebra using Theorem 8.2.

Exercise 2.
Let M, N be KG-modules. Prove that:

(a) $M \cong (M^*)^*$ as KG-modules (in a natural way);

(b) $M^* \oplus N^* \cong (M \oplus N)^*$ and $M^* \otimes_K N^* \cong (M \otimes_K N)^*$ as KG-modules (in a natural way);

(c) M is simple, resp. indecomposable, resp. semisimple, if and only if M^* is simple, resp. semisimple, resp. indecomposable.

Exercise 3.
Let $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ be a s.e.s. of KG-modules. Prove that if $M \cong L \oplus N$, then the s.e.s. splits.

[Hint: Consider the exact sequence induced by the functor $\mathrm{Hom}_{KG}(N, -)$ and use the fact that the modules considered are all finite-diemensional.]

B. Exercises to hand in.

Exercise 4.
Let M, N be KG-modules. Prove that:

(a) the map

$$\begin{array}{rccl} \theta := \theta_{M,N}: & M^* \otimes_K N & \longrightarrow & \mathrm{Hom}_K(M,N) \\ & f \otimes n & \mapsto & \theta(f\otimes n): M \longrightarrow N, m \mapsto \theta(f\otimes n)(m) = f(m)n \end{array}$$

is a K-isomorphism;

(b) Tr_M is a KG-homomorohism and $\mathrm{Tr}_M \circ \theta^{-1}_{M,M}$ coincides with the ordinary trace of matrices;

(c) $M \mid M \otimes_K M^* \otimes_K M$ and if $\mathrm{char}(K) \mid \dim_K(M)$, then $M \oplus M \mid M \otimes_K M^* \otimes_K M$. (This is more challenging!)

Exercise 5.
Prove that $\mathrm{Coind}_{\{1\}}^G(K) \cong (KG)^*$ as KG-modules by defining an explicit KG-isomorphism.
[Warning: with their KG-module structures $\mathrm{Coind}_{\{1\}}^G(K)$ and $(KG)^*$ are isomorphic but not equal!]

Exercise 6.
Let U, V, W be KG-modules. Prove that there are isomorphisms of KG-modules:

(i) $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(U, V^* \otimes_K W)$; and

(ii) $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W) \cong \mathrm{Hom}_{KG}(U, \mathrm{Hom}_K(V, W))$.

Exercise 7 (Optional Exercise).
Investigate whether the statements of Exercise 2, Exercise 4, and Exercise 5 can be generalised to the case in which K is an arbitrary commutative ring and the KG-modules are free of finite rank when seen as K-modules.