Throughout, all rings are assumed to be *associative rings with* 1, and modules are assumed to be *left* modules.

A. Exercises for the tutorial.

Exercise 1.
Let R be a semisimple ring. Prove the following statements.

(a) Every non-zero left ideal I of R is generated by an **idempotent** of R, in other words $\exists\, e \in R$ such that $e^2 = e$ and $I = Re$.
[Hint: choose a complement I' for I, so that $R^\circ = I \oplus I'$ and write $1 = e + e'$ with $e \in I$ and $e' \in I'$. Prove that $I = Re$.]

(b) If I is a non-zero left ideal of R, then every morphism in $\mathrm{Hom}_R(I, R^\circ)$ is given by right multiplication with an element of R.

(c) If $e \in R$ is an idempotent, then $\mathrm{End}_R(Re) \cong (eRe)^{\mathrm{op}}$ (the opposite ring) as rings via the map $f \mapsto ef(e)e$. In particular $\mathrm{End}_R(R^\circ) \cong R^{\mathrm{op}}$ via $f \mapsto f(1)$.

(d) A left ideal Re generated by an idempotent e of R is minimal (i.e. simple as an R-module) if and only if eRe is a division ring.
[Hint: Use Schur's Lemma.]

(e) Every simple left R-module is isomorphic to a minimal left ideal in R, i.e. a simple R-submodule of R°.

Exercise 2.
Let K be a commutative ring and A be a K-algebra.

(a) Prove that $Z(A)$ is a K-subalgebra of A;

(b) Prove that if K is a field and $A \neq 0$, then $K \longrightarrow Z(A), \lambda \mapsto \lambda 1_A$ is an injective K-homomorphism.

(c) Prove that if $A = M_n(K)$ ($n \in \mathbb{Z}_{>0}$), then $Z(A) = KI_n$, i.e. the K-subalgebra of scalar matrices.
[Hint: $\forall\, 1 \leq i, j \leq n$ denote by $E_{i,j}$ the elementary matrix with (i, j)-th entry equal to 1 (and all other entries equal to zero). Remember that $E_{p,q}E_{s,t} = E_{p,t}$ if $q = s$ and is 0 otherwise.]

(d) Assume A is the algebra of 2×2 upper-triangular matrices over K. Prove that

$$Z(A) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in K \right\}.$$

B. Exercises to hand in.

Exercise 3.

Let K be a field and let $A \neq 0$ be a finite-dimensional K-algebra. The aim of this exercise is to prove that $J(A)$ is the unique maximal nilpotent left ideal of A and $J(Z(A)) = J(A) \cap Z(A)$. Proceed as follows:

(a) Prove that there exists $n \in \mathbb{Z}_{>0}$ such that $J(A)^n = J(A)^{n+1}$.
(Hint: consider dimensions.)

(b) Apply Nakayama's Lemma to deduce that $J(A)^n = 0$ and conclude that $J(A)$ is nilpotent.

(c) Prove that if I is an arbitrary nilpotent left ideal of A, then $I \subseteq J(A)$.
(Hint: here you should see $J(A)$ as the intersection of the annihilators of the simple A-modules.)

(d) Use the nilpotency of the Jacobson radical (of both A and $Z(A)$) to prove that

$$J(Z(A)) = J(A) \cap Z(A)\,.$$

Exercise 4.

(a) Let G be a finite group and K be a commutative ring. Verify that the regular representation ρ_{reg} corresponds to the regular KG-module KG°.

(b) Let $G := C_2 \times C_2$ be the Klein-four group and let $K = \overline{K}$ be an algebraically closed field of characteristic 2.

(i) Prove that $KG \cong K[X, Y]/(X^2, Y^2)$ as K-algebras.
(Note: $K[X, Y]$ stands for the commutative polynomial K-algebra in the variables X and Y, i.e. $XY = YX$ in $K[X, Y]$.)

(ii) Compute $J\left(K[X, Y]/(X^2, Y^2)\right)$ and $|\mathcal{M}(K[X, Y]/(X^2, Y^2))|$, and describe all simple KG-modules.
(Hint: Do not forget that you can consider K-dimensions!)

Exercise 5.

The aim of this exercise is to prove that if K is a field of positive characteristic p and G is a p-group, then $I(KG) = J(KG)$. Proceed as indicated:

(a) Recall that an ideal I of a ring R is called a **nil ideal** if each element of I is nilpotent. Accept the following result: if I is a nil left ideal in a left Artinian ring R then I is nilpotent.

(b) Prove that $g - 1$ is a nilpotent element for each $g \in G \setminus \{1\}$ and deduce that $I(KG)$ is a nil ideal of KG.

(c) Deduce from (a) and (b) that $I(KG) \subseteq J(KG)$ using Exercise 3.

(d) Conclude that $I(KG) = J(KG)$ using Proposition-Definition 10.7.