REPRESENTATION THEORY — EXERCISE SHEET 1 **TU KAISERSLAUTERN**
JUN.-PROF. DR. CAROLINE LASSUEUR **FB MATHEMATIK**
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Throughout, R denotes a ring, and, unless otherwise stated, all rings are assumed to be *associative rings with* 1, and modules are assumed to be *left* modules.

***A. Exercises for the tutorial*.**

EXERCISE 1.
Prove that if $(R, +, \cdot)$ is a ring, then $R^\circ := R$ itself may be seen as an R-module via left multiplication in R, i.e. where the external composition law is given by

$$R \times R^\circ \longrightarrow R^\circ, (r, m) \mapsto r \cdot m \,.$$

We call R° the **regular** R-module.
Prove that:

(a) the R-submodules of R° are prescisely the left ideals of R;

(b) $I \triangleleft R$ is a maximal left ideal of $R \Leftrightarrow R^\circ/I$ is a simple R-module, and $I \triangleleft R$ is a minimal left ideal of $R \Leftrightarrow I$ is simple when regarded as an R-submodule of R°.

EXERCISE 2.

(a) Give a concrete example of an R-module which is indecomposable but not simple.

(b) Prove that any submodule and any quotient of a semisimple module is again semisimple.

EXERCISE 3.
Prove Part (iii) of Fitting's Lemma.

B. Exercises to hand in.

Exercise 4.

(a) Let p be a prime number and $R := \{\frac{a}{b} \in \mathbb{Q} \,\big|\, p \nmid b\}$. Prove that $R \setminus R^\times = \{\frac{a}{b} \in R \,\big|\, p|a\}$ and deduce that R is local.

(b) Let K be a field and let $R := \left\{A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \dots & a_{n-1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_1 \end{pmatrix} \in M_n(K)\right\}$.
Prove that $R \setminus R^\times = \{A \in R \mid a_1 = 0\}$ and deduce that R is local.

Exercise 5.

(a) Prove that any simple R-module may be seen as a simple $R/J(R)$-module.

(b) Conversely, prove that any simple $R/J(R)$-module may be seen as a simple R-module. [Hint: use a change of the base ring via the canonical morphism $R \longrightarrow R/J(R)$.]

(c) Deduce that R and $R/J(R)$ have the same simple modules.

Exercise 6.

(a) Let K be a field and let A be the K-algebra $\left\{\begin{pmatrix} a_1 & a \\ 0 & a_1 \end{pmatrix} \mid a_1, a \in K\right\}$. Consider the A-module $V := K^2$, where A acts by left matrix multiplication. Prove that:

(1) $\{\binom{x}{0} \mid x \in K\}$ is a simple A-submodule of V; but

(2) V is not semisimple.

(b) Prove that $J(\mathbb{Z}) = 0$ and find an example of a $\mathbb{Z}$-module which is not semisimple.