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Due date: – | WS 2020/21

This sheet is a revision sheet on modules and algebras. These exercises are to be discussed with the assistant during the tutorial on the 29th of October.

Throughout, R and S denote rings. Unless otherwise stated, all rings are assumed to be *associative rings with* 1, and modules are assumed to be *left* modules.

Exercise 1 (**Change of the base ring**).
Prove that if $\varphi : S \longrightarrow R$ is a ring homomorphism, then every R-module M can be endowed with the structure of an S-module with external composition law given by

$$\cdot : S \times M \longrightarrow M, (s, m) \mapsto s \cdot m := \varphi(s) \cdot m\,.$$

[Hint: use Definition A.1(1) rather than the module axioms in Example A.4(a).]

Exercise 2.
Prove that if K is a field, then any K-algebra of dimension 2 is commutative.

[Hint: consider a K-basis containing the 1 element.]

Exercise 3.
Assume that R is a commutative ring

(a) Let M, N be R-modules. Prove that the abelian group $\mathrm{Hom}_R(M, N)$ is a left R-module for the external composition law defined by

$$(rf)(m) := f(rm) = rf(m) \qquad \forall\, r \in R,\ \forall f \in \mathrm{Hom}_R(M, N),\ \forall m \in M\,.$$

(b) Let A be an R-algebra and M be an A-module. Prove that $\mathrm{End}_R(M)$ and $\mathrm{End}_A(M)$ are R-algebras.

Exercise 4.
Assume R is a commutative ring and I is an ideal of R. Let M be a left R-module. Prove that there is an isomorphism of left R-modules

$$R/I \otimes_R M \cong M/IM\,.$$