JUN.-PROF. DR. CAROLINE LASSUEUR **FB MATHEMATIK**
DR. NIAMH FARRELL
Due date: **Tuesday, 10th of December 2019, 6 p.m.** WS 2019/20

Throughout G denotes a finite group and K a commutative ring. All KG-modules considered are assumed to be finitely generated and free as K-modules.

EXERCISE 21. (a) (**Adjunction.**) Let R and S be rings (i.e. associative with 1). Let M be a left R-module, let N be a left S-module and let W be an (S,R)-bimodule. Prove that

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(W,N)) \cong \mathrm{Hom}_S(W \otimes_R M, N)\,.$$

(b) Verify that the first isomorphism in the statement of Frobenius reciprocity is a particular case of (a).

(c) Prove Proposition 20.11(b) using adjunction and the fact that induction and coinduction coincide.

EXERCISE 22.
Let $L \leq H \leq G$. Prove that:

(a) $\mathrm{Coind}_{\{1\}}^G(K) \cong (KG)^*$ as KG-modules (i.e. exhibit a concret isomorphism);

(b) (transitivity of induction) if M is a KL-module, then $M{\uparrow}_L^G = (M{\uparrow}_L^H){\uparrow}_H^G$;

(c) if M is a KH-module, then $(M^*){\uparrow}_H^G \cong (M{\uparrow}_H^G)^*$; and

(d) if M is a KG-module, then $(M^*){\downarrow}_H^G \cong (M{\downarrow}_H^G)^*$.

EXERCISE 23.
Let K be a field.

(a) Let U, V, W be KG-modules. Prove that there isomorphisms of KG-modules:

(i) $\mathrm{Hom}_K(U \otimes_K V, W) \cong \mathrm{Hom}_K(U, V^* \otimes_K W)$; and

(ii) $\mathrm{Hom}_{KG}(U \otimes_K V, W) \cong \mathrm{Hom}_{KG}(U, V^* \otimes_K W) \cong \mathrm{Hom}_{KG}(U, \mathrm{Hom}_K(V,W))$.

(b) Prove Proposition 20.11(b) using Proposition 20.11(a).

Exercise 24. (a) Let $H, L \leq G$. Prove that the set of (H, L)-double cosets is in bijection with the set of orbits $H\backslash(G/L)$, and also with the set of orbits $(H\backslash G)/L$ under the mappings

$$HgL \mapsto H(gL) \in H\backslash(G/L)$$

$$HgL \mapsto (Hg)L \in (H\backslash G)/L.$$

This justifies the notation $H\backslash G/L$ for the set of (H, L)-double cosets.

(b) Let $G = S_3$. Consider $H = L := S_2 = \{\mathrm{Id}, (1\ 2)\}$ as a subgroup of S_3. Prove that

$$[S_2\backslash S_3/S_2] = \{\mathrm{Id}, (1\ 2\ 3)\}$$

while

$$S_2\backslash S_3/S_2 = \{\,\{\mathrm{Id}, (1\ 2)\}, \{(1\ 2\ 3), (1\ 3\ 2), (1\ 3), (2\ 3)\}\}\,.$$