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Due date: **Tuesday, 3rd of December 2019, 6 p.m.** WS 2019/20

Throughout G denotes a finite group and K a commutative ring. All KG-modules considered are assumed to be finitely generated and free as K-modules.

Exercise 17.
Assume K is a field. Let M, N be KG-modules. Prove that

(a) If $\rho : M \to N$ is an injective (resp. surjective) KG-homomorphism, then $\rho^* : N^* \to M^*$ is surjective (resp. injective).
Conclude that if $X \subseteq N$ is a KG-submodule, there exists a KG-submodule $Y \subseteq N^*$ such that $Y \cong (N/X)^*$ and $N^*/Y \cong X^*$.

(b) $M \cong (M^*)^*$ as KG-modules.

(c) $M^* \oplus N^* \cong (M \oplus N)^*$ and $M^* \otimes_K N^* \cong (M \otimes_K N)^*$ as KG-modules.

(d) M is simple, resp. indecomposable, if and only if M^* is simple, resp. indecomposable.

Exercise 18.
Assume K is a field and let M be a KG-module. Prove that:

(a) Tr_M is a KG-homomorohism and $\mathrm{Tr}_M \circ \theta_{M,M}^{-1}$ coincides with the ordinary trace of matrices;

(b) $M \mid M \otimes_K M^* \otimes_K M$;

(c) if $p \mid \dim_K(M)$, then $M \oplus M \mid M \otimes_K M^* \otimes_K M$.

Exercise 19.
Prove that:

(a) $(KG)^G = \langle \sum_{g \in G} g \rangle_K$;

(d) If M and N are KG-modules, then $(M \otimes_K N)_G \cong M \otimes_{KG} N$.

Exercise 20.
Assume K is a field and let $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ be a s.e.s. of KG-modules. Prove that if $M \cong L \oplus N$, then the s.e.s. splits.
[Hint: Consider the exact sequence induced by $\mathrm{Hom}_{KG}(N, -)$ (i.e. as in Proposition 4.3(a)) and use the fact that the modules considered are all finite-diemensional.]