yields

$$|C_G(x)| = \frac{|G|}{|[x]|}\,.$$

Moreover, a function $f : G \longrightarrow \mathbb{C}$ which is constant on each conjugacy class of G, i.e. such that $f(gxg^{-1}) = f(x)\ \forall\, g, x \in G$, is called a **class function** (on G).

Lemma 7.8

Characters are class functions.

Proof: Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Again, because by the properties of the trace we have $\mathrm{Tr}(\beta \circ \gamma) = \mathrm{Tr}(\gamma \circ \beta)$ for all $\mathbb{C}$-endomorphisms β, γ of V (GDM !), it follows that for all $g, x \in G$,

$$\begin{aligned}\chi_V(gxg^{-1}) = \mathrm{Tr}\left(\rho_V(gxg^{-1})\right) &= \mathrm{Tr}\left(\rho_V(g)\rho_V(x)\rho_V(g)^{-1}\right)\\ &= \mathrm{Tr}\big(\rho_V(x)\underbrace{\rho_V(g)^{-1}\rho_V(g)}_{=\mathrm{Id}_V}\big) = \mathrm{Tr}\left(\rho_V(x)\right) = \chi_V(x)\,.\end{aligned}$$

■

Exercise 7.9

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Prove the following statements.

(a) If $g \in G$ is conjugate to g^{-1}, then $\chi_V(g) \in \mathbb{R}$.

(b) If $g \in G$ is an element of order 2, then $\chi_V(g) \in \mathbb{Z}$ and $\chi_V(g) \equiv \chi_V(1) \pmod 2$.

Exercise 7.10 (The dual representation / the dual character)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation.

(a) Prove that:

(i) the dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is endowed with the structure of a $\mathbb{C}G$-module via

$$\begin{array}{ccc} G \times V^* & \longrightarrow & V^* \\ (g, f) & \mapsto & g.f \end{array}$$

where $(g.f)(v) := f(g^{-1}v)\ \forall\, v \in V$;

(ii) the character of the associated $\mathbb{C}$-representation ρ_{V^*} is then $\chi_{V^*} = \overline{\chi_V}$; and

(iii) if ρ_V decomposes as a direct sum $\rho_{V_1} \oplus \rho_{V_2}$ of two subrepresentations, then ρ_{V^*} is equivalent to $\rho_{V_1^*} \oplus \rho_{V_2^*}$.

(b) Determine the duals of the 3 irreducible representations of S_3 given in Example 2(d).

8 Orthogonality of Characters

We are now going to make use of results from the linear algebra (GDM) on the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on G in order to develop further fundamental properties of characters.

Notation 8.1

We let $\mathcal{F}(G,\mathbb{C}) := \{f : G \longrightarrow \mathbb{C} \mid f \text{ function}\}$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on G. Clearly $\dim_{\mathbb{C}} \mathcal{F}(G,\mathbb{C}) = |G|$ because $\{\delta_g : G \longrightarrow \mathbb{C}, h \mapsto \delta_{gh} \mid g \in G\}$ is a $\mathbb{C}$-basis (see GDM). Set $\mathcal{C}l(G) := \{f \in \mathcal{F}(G,\mathbb{C}) \mid f \text{ is a class function}\}$. This is clearly a $\mathbb{C}$-subspace of $\mathcal{F}(G,\mathbb{C})$, called the **space of class functions on** G.

Exercise 8.2

Find a $\mathbb{C}$-basis of $\mathcal{C}l(G)$ and deduce that $\dim_{\mathbb{C}} \mathcal{C}l(G) = |C(G)|$.

Proposition 8.3

The binary operation

$$\begin{array}{rccl} \langle\, ,\, \rangle_G : & \mathcal{F}(G,\mathbb{C}) \times \mathcal{F}(G,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & (f_1, f_2) & \mapsto & \langle f_1, f_2\rangle_G := \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} \end{array}$$

is a scalar product on $\mathcal{F}(G,\mathbb{C})$.

Proof: It is straightforward to check that $\langle\, ,\, \rangle_G$ is sesquilinear and Hermitian (Exercise on Sheet 3); it is positive definite because for every $f \in \mathcal{F}(G,\mathbb{C})$,

$$\langle f, f\rangle_G = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{f(g)} = \frac{1}{|G|}\sum_{g\in G} \underbrace{|f(g)|^2}_{\in\mathbb{R}_{\geqslant 0}} \geqslant 0$$

and moreover $\langle f, f\rangle_G = 0$ if and only if $f = 0$. ■

Remark 8.4

Obviously, the scalar product $\langle\, ,\, \rangle_G$ restricts to a scalar product on $\mathcal{C}l(G)$. Moreover, if f_2 is a character of G, then by Property 7.5(d) we can write

$$\langle f_1, f_2\rangle_G = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)} = \frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g^{-1})\,.$$

The next theorem is the third key result of this lecture. It tells us that the irreducible characters of a finite group form an orthonormal system in $\mathcal{C}l(G)$ with respect to the scalar product $\langle\, ,\, \rangle_G$.

Theorem 8.5 (1st Orthogonality Relations)

If $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ are two irreducible $\mathbb{C}$-representations affording the characters χ_V and χ_W respectively, then

$$\langle \chi_V, \chi_W\rangle_G = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\chi_W(g^{-1}) = \begin{cases} 1 & \text{if } \rho_V \sim \rho_W, \\ 0 & \text{if } \rho_V \not\sim \rho_W. \end{cases}$$

Proof: Choose ordered $\mathbb{C}$-bases $E := (e_1, \ldots, e_n)$ and $F := (f_1, \ldots, f_m)$ of V and W respectively. Then for each $g \in G$ write $Q(g) := \big(\rho_V(g)\big)_E$ and $P(g) := \big(\rho_W(g)\big)_F$. If $\rho_V \not\sim \rho_W$ compute

$$\begin{aligned}\langle \chi_V, \chi_W \rangle_G = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_W(g^{-1}) &= \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\left(Q(g)\right)\mathrm{Tr}\left(P(g^{-1})\right)\\ &= \frac{1}{|G|}\sum_{g\in G}\Big(\sum_{i=1}^{n} Q(g)_{ii}\Big)\Big(\sum_{j=1}^{m} P(g^{-1})_{jj}\Big)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\underbrace{\frac{1}{|G|}\sum_{g\in G} Q(g)_{ii}P(g^{-1})_{jj}}_{=0 \text{ by (a) of Schur's Relations}} = 0\end{aligned}$$

and similarly if $\rho_V \sim \rho_W$, then by Lemma 7.3, we may assume w.l.o.g. that $W = V$, so $P = Q$ and we obtain

$$\langle \chi_V, \chi_V \rangle_G = \sum_{i=1}^{n}\sum_{j=1}^{m}\underbrace{\frac{1}{|G|}\sum_{g\in G} Q(g)_{ii}Q(g^{-1})_{jj}}_{=\frac{1}{n}\delta_{ij}\delta_{ij} \text{ by (b) of Schur's Relations}} = \sum_{i=1}^{n}\frac{1}{n} = 1\,.$$

■

9 Consequences of the 1st Orthogonality Relations

In this section we use the 1st Orthogonality Relations in order to deduce a series of fundamental properties of the (irreducible) characters of finite groups.

Corollary 9.1 (*Linear independence*)

The irreducible characters of G are $\mathbb{C}$-linearly independent.

Proof: Assume $\sum_{i=1}^{s}\lambda_i\chi_i = 0$, where $\chi_1, \dots, \chi_s$ are pairwise distinct irreducible characters of G, $\lambda_1, \dots, \lambda_s \in \mathbb{C}$ and $s \in \mathbb{Z}_{>0}$. Then the 1st Orthogonality Relations yield

$$0 = \Big\langle \sum_{i=1}^{s}\lambda_i\chi_i, \chi_j \Big\rangle_G = \sum_{i=1}^{s}\lambda_i \underbrace{\langle \chi_i, \chi_j\rangle_G}_{=\delta_{ij}} = \lambda_j$$

for each $1 \leqslant j \leqslant s$. The claim follows. ■

Corollary 9.2 (*Finiteness*)

There are at most $|C(G)|$ irreducible characters of G. In particular, there are only a finite number of them.

Proof: By Corollary 9.1 the irreducible characters of G are $\mathbb{C}$-linearly independent. By Lemma 7.8 irrreducible characters are elements of the $\mathbb{C}$-vector space $\mathcal{Cl}(G)$. Therefore there exists at most $\dim_{\mathbb{C}}\mathcal{Cl}(G) = |C(G)| < \infty$ of them. ■

Corollary 9.3 (*Multiplicities*)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let $\rho_V = \rho_{V_1} \oplus \cdots \oplus \rho_{V_s}$ be a decomposition of ρ_V into irreducible subrepresentations. Then the following assertions hold.

(a) If $\rho_W : G \longrightarrow \mathrm{GL}(W)$ is an irreducible $\mathbb{C}$-representation of G, then the multiplicity of ρ_W in $\rho_{V_1} \oplus \cdots \oplus \rho_{V_s}$ is equal to $\langle \chi_V, \chi_W \rangle_G$.

(b) This multiplicity is independent of the choice of the chosen decomposition of ρ_V into irreducible subrepresentations.

Proof: (a) W.l.o.g., we may assume that we have chosen the labelling such that

$$\rho_V = \rho_{V_1} \oplus \cdots \oplus \rho_{V_l} \oplus \rho_{V_{l+1}} \oplus \cdots \oplus \rho_{V_s},$$

where $\rho_{V_i} \sim \rho_W \ \forall\ 1 \leqslant i \leqslant l$ and $\rho_{V_j} \nsim \rho_W \ \forall\ l+1 \leqslant j \leqslant s$. Thus $\chi_{V_i} = \chi_W \ \forall\ 1 \leqslant i \leqslant l$ by Lemma 7.3. Therefore the 1st Orthogonality Relations yield

$$\langle \chi_V, \chi_W \rangle_G = \sum_{i=1}^{l} \langle \chi_{V_i}, \chi_W \rangle_G + \sum_{j=l+1}^{s} \langle \chi_{V_j}, \chi_W \rangle_G = \sum_{i=1}^{l} \underbrace{\langle \chi_W, \chi_W \rangle_G}_{=1} + \sum_{j=l+1}^{s} \underbrace{\langle \chi_{V_j}, \chi_W \rangle_G}_{=0} = l.$$

(b) Obvious, since $\langle \chi_V, \chi_W \rangle_G$ depends only on V and W, but not on the chosen decomposition. ■

We can now prove that the converse of Lemma 7.3 holds.

Corollary 9.4 (*Equality of characters*)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be $\mathbb{C}$-representations with characters χ_V and χ_W respectively. Then,

$$\chi_V = \chi_W \quad \Leftrightarrow \quad \rho_V \sim \rho_W.$$

Proof: "$\Leftarrow$": The sufficient condition is the statement of Lemma 7.3.

"$\Rightarrow$": To prove the necessary condition decompose ρ_V and ρ_W into direct sums of irreducible subrepresentations

$$\rho_V = \underbrace{\rho_{V_{1,1}} \oplus \cdots \oplus \rho_{V_{1,m_1}}}_{\text{all } \sim \rho_{V_1}} \oplus \cdots \oplus \underbrace{\rho_{V_{s,1}} \oplus \cdots \oplus \rho_{V_{s,m_s}}}_{\text{all } \sim \rho_{V_s}},$$

$$\rho_W = \underbrace{\rho_{W_{1,1}} \oplus \cdots \oplus \rho_{W_{1,p_1}}}_{\text{all } \sim \rho_{V_1}} \oplus \cdots \oplus \underbrace{\rho_{W_{s,1}} \oplus \cdots \oplus \rho_{W_{s,p_s}}}_{\text{all } \sim \rho_{V_s}},$$

where $m_i, p_i \geqslant 0$ for all $1 \leqslant i \leqslant s$ and the ρ_{V_i}'s are pairwise non-equivalent irreducible $\mathbb{C}$-representations of G. (Some of the m_i, p_i's may be zero!) Now, as we assume that $\chi_V = \chi_W$, for each $1 \leqslant i \leqslant s$ Corollary 9.3 yields

$$m_i = \langle \chi_V, \chi_{V_i} \rangle_G = \langle \chi_W, \chi_{V_i} \rangle_G = p_i,$$

hence $\rho_V \sim \rho_W$. ■

Corollary 9.5 (*Irreducibility criterion*)

A $\mathbb{C}$-representation $\rho_V : G \longrightarrow \mathrm{GL}(V)$ is irreducible if and only if $\langle \chi_V, \chi_V \rangle_G = 1$.

Proof: "$\Rightarrow$": holds by the 1st Orthogonality Relations.

"$\Leftarrow$": As in the previous proof, write

$$\rho_V = \underbrace{\rho_{V_{1,1}} \oplus \cdots \oplus \rho_{V_{1,m_1}}}_{\text{all } \sim \rho_{V_1}} \oplus \cdots \oplus \underbrace{\rho_{V_{s,1}} \oplus \cdots \oplus \rho_{V_{s,m_s}}}_{\text{all } \sim \rho_{V_s}},$$

where $m_i \geqslant 1$ for all $1 \leqslant i \leqslant s$ and the ρ_{V_i}'s are pairwise non-equivalent irreducible $\mathbb{C}$-representations of G. Then, using the assumption, the sesquilinearity of the scalar product and the 1st Orthogonality Relations, we obtain that

$$1 = \langle \chi_V, \chi_V \rangle_G = \sum_{i=1}^{s} m_i^2 \underbrace{\langle \chi_{V_i}, \chi_{V_i} \rangle_G}_{=1} = \sum_{i=1}^{s} m_i^2 \,.$$

Hence, w.l.o.g. we may assume that $m_1 = 1$ and $m_i = 0 \;\forall\; 2 \leqslant i \leqslant s$, so that $\rho_V = \rho_{V_1}$ is irreducible. ■

Theorem 9.6

The set $\mathrm{Irr}(G)$ is an orthonormal $\mathbb{C}$-basis (w.r.t. $\langle\, , \,\rangle_G$) of the $\mathbb{C}$-vector space $\mathcal{C}l(G)$ of class functions on G.

Proof: We already know that $\mathrm{Irr}(G)$ is a $\mathbb{C}$-linearly independent set and also that it forms an orthonormal system of $\mathcal{C}l(G)$ w.r.t. $\langle\, , \,\rangle_G$. Hence it remains to prove that $\mathrm{Irr}(G)$ generates $\mathcal{C}l(G)$ as a $\mathbb{C}$-vector space. So let $X := \langle \mathrm{Irr}(G) \rangle_{\mathbb{C}}$ be the $\mathbb{C}$-subspace of $\mathcal{C}l(G)$ generated by $\mathrm{Irr}(G)$. It follows that

$$\mathcal{C}l(G) = X \oplus X^{\perp}$$

where $X^{\perp}$ denotes the orthogonal of X with respect to the scalar product $\langle\, , \,\rangle_G$ (see GDM). Thus it is enough to prove that $X^{\perp} = 0$. So let $f \in X^{\perp}$, set $\check{f} := \sum_{g \in G} \overline{f(g)} g \in \mathbb{C}G$ and we prove the following assertions:

(1) $\check{f} \in Z(\mathbb{C}G)$ (the centre of $\mathbb{C}G$): let $h \in G$ and compute

$$h\check{f}h^{-1} = \sum_{g \in G} \overline{f(g)} hg \cdot h^{-1} \overset{s := hgh^{-1}}{=} \sum_{s \in G} \underbrace{\overline{f(h^{-1}sh)}}_{=f(s)} s = \sum_{s \in G} \overline{f(s)} s = \check{f} \,.$$

Hence $h\check{f} = \check{f}h$ and this equality extends by $\mathbb{C}$-linearity to the whole of $\mathbb{C}G$, so that $\check{f} \in Z(\mathbb{C}G)$.

(2) If V is a simple $\mathbb{C}G$-module with character χ_V, then the external multiplication by $\check{f}$ on V is scalar multiplication by $\frac{|G|}{\dim_{\mathbb{C}} V} \langle \chi_V, f \rangle_G \in \mathbb{C}$: first notice that the external multiplication by $\check{f}$ on V, i.e. the map

$$\check{f} \cdot - : V \longrightarrow V, v \mapsto \check{f} \cdot v$$

is $\mathbb{C}G$-linear (i.e. an element of $\mathrm{End}_{\mathbb{C}G}(V)$). Indeed, for each $x \in \mathbb{C}G$ and each $v \in V$ we have

$$\check{f} \cdot (x \cdot v) = (\check{f}x) \cdot v = (x\check{f}) \cdot v = x \cdot (\check{f} \cdot v)$$

because $\check{f} \in Z(\mathbb{C}G)$. Therefore, by Schur's Lemma, there exists a scalar $\lambda \in \mathbb{C}$ such that $\check{f} \cdot - = \lambda \,\mathrm{Id}_V$. Now, setting $n := \dim_{\mathbb{C}}(V)$, we have

$$\lambda = \frac{1}{n} \mathrm{Tr}(\lambda \,\mathrm{Id}_V) = \frac{1}{n} \mathrm{Tr}(\check{f} \cdot -) = \frac{1}{n} \sum_{g \in G} \overline{f(g)} \underbrace{\mathrm{Tr}\left(\text{mult. by } g \text{ on } V\right)}_{=\chi_V(g)} = \frac{1}{n} \sum_{g \in G} \overline{f(g)} \chi_V(g) = \frac{|G|}{n} \langle \chi_V, f \rangle_G \,.$$

(3) If V is a simple $\mathbb{C}G$-module with character χ_V, then the external multiplication by $\check{f}$ on V is zero: indeed, $\langle \chi_V, f \rangle_G = 0$ because $f \in X^{\perp}$ and the claim follows from (2).

(4) $f = 0$: indeed, as the external multiplication by $\check{f}$ is zero on every simple $\mathbb{C}G$-module, it is zero on every $\mathbb{C}G$-module, because any $\mathbb{C}G$-module can be decomposed as the direct sum of simple submodules

by the Corollary to Maschke's Theorem. In particular, the external multiplication by $\check{f}$ is zero on $\mathbb{C}G$. Hence

$$0 = \check{f} \cdot 1_{\mathbb{C}G} = \check{f} = \sum_{g \in G} \overline{f(g)} g$$

and we obtain that $\overline{f(g)} = 0$ for each $g \in G$ because G is a $\mathbb{C}$-basis of $\mathbb{C}G$. But then $f(g) = 0$ for each $g \in G$ and it follows that $f = 0$. ■

Corollary 9.7

The number of pairwise distinct irreducible characters of G is equal to the number of conjugacy classes of G. In other words,

$$|\operatorname{Irr}(G)| = |C(G)|.$$

Proof: By Theorem 9.6 the set $\operatorname{Irr}(G)$ is a $\mathbb{C}$-basis of the $\mathbb{C}$-vector space $\mathcal{C}l(G)$ of class functions on G. Hence,

$$|\operatorname{Irr}(G)| = \dim_{\mathbb{C}} \mathcal{C}l(G) = |C(G)|$$

where the second equality holds by Exercise 8.2. ■

Corollary 9.8

Let $f \in \mathcal{C}l(G)$. Then the following assertions hold:

(a) $f = \sum_{\chi \in \operatorname{Irr}(G)} \langle f, \chi \rangle_G \chi$;

(b) $\langle f, f \rangle_G = \sum_{\chi \in \operatorname{Irr}(G)} \langle f, \chi \rangle_G^2$;

(c) f is a character $\iff \langle f, \chi \rangle_G \in \mathbb{Z}_{\geqslant 0} \;\; \forall\, \chi \in \operatorname{Irr}(G)$; and

(d) $f \in \operatorname{Irr}(G) \iff f$ is a character and $\langle f, f \rangle_G = 1$.

Proof: (a)+(b) hold for any orthonormal basis with respect to a given scalar product. (GDM!)

(c) '$\Rightarrow$': If f is a character, then by Corollary 9.3 the complex number $\langle f, \chi \rangle_G$ is the multiplicity of χ as a constituent of f for each $\chi \in \operatorname{Irr}(G)$, hence a non-negative integer.

'$\Leftarrow$': If for each $\chi \in \operatorname{Irr}(G)$, $\langle f, \chi \rangle_G =: m_\chi \in \mathbb{Z}_{\geqslant 0}$, then f is the character of the representation

$$\rho := \bigoplus_{\chi \in \operatorname{Irr}(G)} \bigoplus_{j=1}^{m_\chi} \rho(\chi)$$

where $\rho(\chi)$ is a $\mathbb{C}$-representation affording the character χ.

(d) The necessary condition is given by the 1st Orthogonality Relations. The sufficient condition follows from (b) and (c). ■

Exercise 9.9

Let V be a $\mathbb{C}G$-module affording the character χ_V. Consider the $\mathbb{C}$-subspace of fixed points under the action of G, that is, $V^G := \{v \in V \mid g \cdot v = v \;\forall\, g \in G\}$. Prove that

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

in two different ways:

1. considering the scalar product of χ_V with the trivial character $\mathbf{1}_G$;
2. seeing V^G as the image of the projector $\pi : V \longrightarrow V, v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v$.