20 Clifford Theory

Clifford theory is a generic term for a series of results relating the representation / character theory of a given group G to that of a normal subgroup $N \trianglelefteq G$ through induction and restriction.

Definition 20.1 (*Conjugate class function / inertia group*)

Let $H \leqslant G$, let $\varphi \in \mathcal{C}l(H)$ and let $g \in G$.

(a) We define ${}^g\varphi \in \mathcal{C}l(gHg^{-1})$ to be the class function on gHg^{-1} defined by

$$ {}^g\varphi : gHg^{-1} \longrightarrow \mathbb{C}, x \mapsto \varphi(g^{-1}xg)\,. $$

(b) The subgroup $\mathcal{I}_G(\varphi) := \{g \in G \mid {}^g\varphi = \varphi\} \leqslant G$ is called the **inertia group** of φ in G.

Exercise 20.2 (*Exercise 26, Sheet 7*)

With the notation of Definition 20.1, prove that:

(a) ${}^g\varphi$ is indeed a class function on gHg^{-1};

(b) $\mathcal{I}_G(\varphi) \leqslant G$ and $H \leqslant \mathcal{I}_G(\varphi) \leqslant N_G(H)$;

(c) for $g, h \in G$ we have ${}^g\varphi = {}^h\varphi \Leftrightarrow h^{-1}g \in \mathcal{I}_G(\varphi) \Leftrightarrow g\mathcal{I}_G(\varphi) = h\mathcal{I}_G(\varphi)$;

(d) if $\rho : H \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of H with character χ, then

$$ {}^g\rho : gHg^{-1} \longrightarrow \mathrm{GL}(V), x \mapsto \rho(g^{-1}xg) $$

is $\mathbb{C}$-representation of gHg^{-1} with character ${}^g\chi$ and ${}^g\chi(1) = \chi(1)$;

(e) if $J \leqslant H$ then ${}^g(\varphi\downarrow^H_J) = ({}^g\varphi)\downarrow^{gHg^{-1}}_{gJg^{-1}}$.

Lemma 20.3

(a) If $H \leqslant G$, $\varphi, \psi \in \mathcal{C}l(H)$ and $g \in G$, then $\langle {}^g\varphi, {}^g\psi \rangle_{gHg^{-1}} = \langle \varphi, \psi \rangle_H$.

(b) If $N \trianglelefteq G$ and $g \in G$, then we have $\psi \in \mathrm{Irr}(N) \Leftrightarrow {}^g\psi \in \mathrm{Irr}(N)$.

(c) If $N \trianglelefteq G$ and ψ is a character of N, then $(\psi\uparrow^G_N)\downarrow^G_N = |\mathcal{I}_G(\psi) : N| \sum_{g \in [G/\mathcal{I}_G(\psi)]} {}^g\psi$.

Proof: (a) Clearly

$$ \begin{aligned} \langle {}^g\varphi, {}^g\psi \rangle_{gHg^{-1}} &= \frac{1}{|gHg^{-1}|} \sum_{x \in gHg^{-1}} {}^g\varphi(x)\overline{{}^g\psi(x)} \\ &= \frac{1}{|H|} \sum_{x \in gHg^{-1}} \varphi(g^{-1}xg)\overline{\psi(g^{-1}xg)} \\ &\overset{y := g^{-1}xg}{=} \frac{1}{|H|} \sum_{y \in H} \varphi(y)\overline{\psi(y)} = \langle \varphi, \psi \rangle_H\,. \end{aligned} $$

(b) As $N \trianglelefteq G$, $gNg^{-1} = N$. Thus, if $\psi \in \mathrm{Irr}(N)$, then on the one hand ${}^g\psi$ is also a character of N by Exercise 20.2(d), and on the other hand it follows from (a) that $\langle {}^g\psi, {}^g\psi\rangle_N = \langle \psi, \psi\rangle_N = 1$. Hence ${}^g\psi$ is an irreducible character of N. Therefore, if ${}^g\psi \in \mathrm{Irr}(N)$, then $\psi = {}^{g^{-1}}({}^g\psi) \in \mathrm{Irr}(N)$, as required.

(c) If $n \in N$ then so does $g^{-1}ng$ $\forall\, g \in G$, hence

$$\psi\uparrow_N^G\downarrow_N^G(n) = \psi\uparrow_N^G(n) = \frac{1}{|N|}\sum_{g\in G}\psi(g^{-1}ng) = \frac{1}{|N|}\sum_{g\in G}{}^g\psi(n) = \frac{|\mathcal{I}_G(\psi)|}{|N|}\sum_{g\in[G/\mathcal{I}_G(\psi)]}{}^g\psi(n).$$ ■

Notation 20.4

Given $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$, we set $\mathrm{Irr}(G \mid \psi) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi\downarrow_N^G, \psi\rangle_N \neq 0\}$.

Theorem 20.5 (Clifford Theory)

Let $N \trianglelefteq G$. Let $\chi \in \mathrm{Irr}(G)$, $\psi \in \mathrm{Irr}(N)$ and set $\mathcal{I} := \mathcal{I}_G(\psi)$. Then the following assertions hold.

(a) If ψ is a constituent of $\chi\downarrow_N^G$, then

$$\chi\downarrow_N^G = e\sum_{g\in[G/\mathcal{I}_G(\psi)]}{}^g\psi\,,$$

where $e = \langle \chi\downarrow_N^G, \psi\rangle_N = \langle \chi, \psi\uparrow_N^G\rangle_G \in \mathbb{Z}_{>0}$ is called the **ramification index** of χ in N (or of ψ in G). In particular, all the constituents of $\chi\downarrow_N^G$ have the same degree.

(b) Induction from $\mathcal{I} = \mathcal{I}_G(\psi)$ to G induces a bijection

$$\begin{array}{rccc} \mathrm{Ind}_{\mathcal{I}}^G: & \mathrm{Irr}(\mathcal{I} \mid \psi) & \longrightarrow & \mathrm{Irr}(G \mid \psi) \\ & \eta & \mapsto & \eta\uparrow_{\mathcal{I}}^G \end{array}$$

preserving ramification indices, i.e. $\langle \eta\downarrow_N^{\mathcal{I}}, \psi\rangle_N = \langle \eta\uparrow_{\mathcal{I}}^G\downarrow_N^{\mathcal{I}}, \psi\rangle_N = e$.

Proof:

(a) By Frobenius reciprocity, $\langle \chi, \psi\uparrow_N^G\rangle_G = \langle \chi\downarrow_N^G, \psi\rangle_N \neq 0$. Thus χ is a constituent of $\psi\uparrow_N^G$ and therefore $\chi\downarrow_N^G$ is a constituent of $\psi\uparrow_N^G\downarrow_N^G$.
Now, if $\eta \in \mathrm{Irr}(N)$ is an arbitrary constituent of $\chi\downarrow_N^G$ (i.e. $\langle \chi\downarrow_N^G, \eta\rangle_N \neq 0$) then by the above, we have

$$\langle \psi\uparrow_N^G\downarrow_N^G, \eta\rangle_N \geqslant \langle \chi\downarrow_N^G, \eta\rangle_N > 0\,.$$

Moroever, by Lemma 20.3(c) the constituents of $\psi\uparrow_N^G\downarrow_N^G$ are preciely $\{{}^g\psi \mid g \in [G/\mathcal{I}_G(\psi)]\}$. Hence η is G-conjugate to ψ. Furthermore, for every $g \in G$ we have

$$\begin{aligned}\langle \chi\downarrow_N^G, {}^g\psi\rangle_N = \frac{1}{|N|}\sum_{h\in N}\chi(h)^g\psi(h^{-1}) &= \frac{1}{|N|}\sum_{h\in N}\chi(h)\psi(g^{-1}h^{-1}g) \\ &\overset{\chi\in\mathcal{C}l(G)}{=} \frac{1}{|N|}\sum_{h\in N}\chi(g^{-1}hg)\psi(g^{-1}h^{-1}g) \\ &\overset{s:=g^{-1}hg\in N}{=} \frac{1}{|N|}\sum_{s\in N}\chi(s)\psi(s^{-1}) = \langle \chi\downarrow_N^G, \psi\rangle_N = e\,.\end{aligned}$$

Therefore, every G-conjugate ${}^g\psi$ ($g \in [G/\mathcal{I}_G(\psi)]$) of ψ occurs as a constituent of $\chi\downarrow_N^G$ with the same multiplicity e. The claim about the degrees is then clear as ${}^g\psi(1) = \psi(1)$ $\forall g \in G$.

(b) **Claim 1:** $\eta \in \mathrm{Irr}(\mathcal{I} \mid \psi) \;\Rightarrow\; \eta\!\uparrow_{\mathcal{I}}^{G} \in \mathrm{Irr}(G|\psi)$.
Since $\mathcal{I} = \mathcal{I}_{\mathcal{I}}(\psi)$, (a) implies that $\eta\!\downarrow_N^{\mathcal{I}} = e'\psi$ with $e' = \langle \eta\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N = \frac{\eta(1)}{\psi(1)} > 0$. Now, let $\chi \in \mathrm{Irr}(G)$ be a constituent of $\eta\!\uparrow_{\mathcal{I}}^{G}$. By Frobenius Reciprocity we have

$$0 \neq \langle \chi, \eta\!\uparrow_{\mathcal{I}}^{G}\rangle_G = \langle \chi\!\downarrow_{\mathcal{I}}^{G}, \eta\rangle_{\mathcal{I}}\,.$$

It follows that $\eta\!\downarrow_N^{\mathcal{I}}$ is a constituent of $\chi\!\downarrow_{\mathcal{I}}^{G}\!\downarrow_N^{\mathcal{I}}$ and

$$e := \langle \chi\!\downarrow_N^{G}, \psi\rangle_N = \langle \chi\!\downarrow_{\mathcal{I}}^{G}\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N \geqslant \langle \eta\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N = e' > 0\,,$$

hence $\chi \in \mathrm{Irr}(G|\psi)$. Moreover, by (a) we have $e = \langle \chi\!\downarrow_N^{G}, {}^{g}\psi\rangle_N \geqslant e'$ for each $g \in G$. Therefore,

$$\chi(1) = e \sum_{g\in[G/\mathcal{I}]} {}^{g}\psi(1) \overset{(a)}{=} e|G:\mathcal{I}|\psi(1) \geqslant e'|G:\mathcal{I}|\psi(1) = |G:\mathcal{I}|\eta(1) = \eta\!\uparrow_{\mathcal{I}}^{G}(1) \geqslant \chi(1)\,.$$

Thus $e = e'$, $\eta\!\uparrow_{\mathcal{I}}^{G} = \chi \in \mathrm{Irr}(G)$, and therefore $\eta\!\uparrow_{\mathcal{I}}^{G} \in \mathrm{Irr}(G|\psi)$.

Claim 2: $\chi \in \mathrm{Irr}(G \mid \psi) \;\Rightarrow\; \exists!\, \eta \in \mathrm{Irr}(\mathcal{I} \mid \psi)$ such that $\langle \chi\!\downarrow_{\mathcal{I}}^{G}, \eta\rangle_{\mathcal{I}} \neq 0$.
Again by (a), as $\chi \in \mathrm{Irr}(G \mid \psi)$, we have $\chi\!\downarrow_N^{G} = e\sum_{g\in[G/\mathcal{I}]} {}^{g}\psi$, where $e = \langle \chi\!\downarrow_N^{G}, \psi\rangle_N \in \mathbb{Z}_{>0}$. Therefore, there exists $\eta \in \mathrm{Irr}(\mathcal{I})$ such that

$$\langle \chi\!\downarrow_{\mathcal{I}}^{G}, \eta\rangle_{\mathcal{I}} \neq 0 \neq \langle \eta\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N$$

because $\chi\!\downarrow_N^{G} = \chi\!\downarrow_{\mathcal{I}}^{G}\!\downarrow_N^{\mathcal{I}}$, so in particular $\eta \in \mathrm{Irr}(\mathcal{I} \mid \psi)$. Hence existence holds and it remains to see that uniqueness holds. Again by Frobenius reciprocity we have $0 \neq \langle \chi, \eta\!\uparrow_{\mathcal{I}}^{G}\rangle_G$. By Claim 1 this forces $\chi = \eta\!\uparrow_{\mathcal{I}}^{G}$ and $\eta\!\downarrow_N^{\mathcal{I}} = e\psi$, so e is also the ramification index of ψ in $\mathcal{I}$.
Now, write $\chi\!\downarrow_{\mathcal{I}}^{G} = \sum_{\lambda\in\mathrm{Irr}(\mathcal{I})} a_\lambda \lambda = \sum_{\lambda\neq\eta} a_\lambda\lambda + a_\eta\eta$ with $a_\lambda \geqslant 0$ for each $\lambda \in \mathrm{Irr}(\mathcal{I})$ and $a_\eta > 0$. It follows that

$$(a_\eta - 1)\eta\!\downarrow_N^{\mathcal{I}} + \sum_{\lambda\neq\eta} a_\lambda\lambda\!\downarrow_N^{\mathcal{I}} = \underbrace{\chi\!\downarrow_N^{G}}_{=e\sum_{g\in[G/\mathcal{I}]} {}^{g}\psi} - \underbrace{\eta\!\downarrow_N^{\mathcal{I}}}_{=e\psi} = e\sum_{g\in[G/\mathcal{I}]\setminus[1]} {}^{g}\psi.$$

Since ψ does not occur in this sum, but occurs in $\eta\!\downarrow_N^{\mathcal{I}}$, the only possibility is $a_\eta = 1$ and $\lambda \notin \mathrm{Irr}(\mathcal{I}|\psi)$ for $\lambda \neq \eta$. Thus η is uniquely determined as the only constituent of $\chi\!\downarrow_{\mathcal{I}}^{G}$ in $\mathrm{Irr}(\mathcal{I} \mid \psi)$.

Finally, Claims 1 and 2 prove that $\mathrm{Ind}_{\mathcal{I}}^{G} : \mathrm{Irr}(\mathcal{I} \mid \psi) \longrightarrow \mathrm{Irr}(G \mid \psi), \eta \mapsto \eta\!\uparrow_{\mathcal{I}}^{G}$ is well-defined and bijective, and the proof of Claim 2 shows that the ramification indices are preserved. ■

Example 13 (*Normal subgroups of index 2*)

Let $N < G$ be a subgroup of index $|G:N| = 2$ ($\Rightarrow N \lhd G$) and let $\chi \in \mathrm{Irr}(G)$, then either

(1) $\chi\!\downarrow_N^{G} \in \mathrm{Irr}(N)$, or

(2) $\chi\!\downarrow_N^{G} = \psi + {}^{g}\psi$ for a $\psi \in \mathrm{Irr}(N)$ and a $g \in G\setminus N$.

Indeed, let $\psi \in \mathrm{Irr}(N)$ be a constituent of $\chi\!\downarrow_N^{G}$. Since $|G:N| = 2$, we have $\mathcal{I}_G(\psi) \in \{N, G\}$. Theorem 20.5 yields the following:

- If $\mathcal{I}_G(\psi) = N$ then $\mathrm{Irr}(\mathcal{I}_G(\psi) \mid \psi) = \{\psi\}$ and $\psi\!\uparrow_N^{G} = \chi$, so that $e = 1$ and we get $\chi\!\downarrow_N^{G} = \psi + {}^{g}\psi$ for any $g \in G\setminus N$.
- If $\mathcal{I}_G(\psi) = G$ then $G/\mathcal{I}_G(\psi) = \{1\}$, so that

$$\chi\!\downarrow_N^{G} = e\psi \qquad \text{with } e = \langle \chi\!\downarrow_N^{G}, \psi\rangle_N = \langle \chi, \psi\!\uparrow_N^{G}\rangle_G\,.$$

Moroever, by Lemma 20.3(c),

$$\psi\uparrow_N^G\downarrow_N^G = |\mathcal{I}_G(\psi):N| \sum_{g\in G/\mathcal{I}_G(\psi)} {}^g\psi = 2\psi\,.$$

Hence

$$2\psi(1) = \psi\uparrow_N^G\downarrow_N^G(1) \geqslant \chi\downarrow_N^G(1) = \chi(1) = e\psi(1) \quad\Rightarrow\quad e \leqslant 2\,.$$

Were $e = 2$ then we would have $2\psi(1) = \psi\uparrow_N^G(1)$, hence $\chi = \psi\uparrow_N^G$ and thus

$$1 = \langle \chi, \psi\uparrow_N^G\rangle_G = \langle \chi\downarrow_N^G, \psi\rangle_N = e = 2$$

a contradiction. Whence $e = 1$, which implies that $\chi\downarrow_N^G \in \mathrm{Irr}(N)$. Moreover, $\psi\uparrow_N^G = \chi + \chi'$ for some $\chi' \in \mathrm{Irr}(G)$ such that $\chi' \neq \chi$.

The following consequence of Clifford's theorem due to N. Itô provides us with a generalisation of the fact that the degrees of the irreducible characters divide the order of the group.

Theorem 20.6 (Itô)

Let $A \leqslant G$ be an abelian subgroup of G and let $\chi \in \mathrm{Irr}(G)$. Then the following assertions hold:

(a) $\chi(1) \leqslant |G:A|$; and

(b) if $A \trianglelefteq G$, then $\chi(1) \,\big|\, |G:A|$.

Proof:

(a) Exercise 27, Sheet 7.

(b) Let $\psi \in \mathrm{Irr}(A)$ be a constituent of $\chi\downarrow_A^G$, so that in other words $\chi \in \mathrm{Irr}(G \mid \psi)$. By Theorem 20.5(b) there exists $\eta \in \mathrm{Irr}(\mathcal{I}_G(\psi) \mid \psi)$ such that $\chi = \eta\uparrow_{\mathcal{I}_G(\psi)}^G$ and $\eta\downarrow_A^{\mathcal{I}_G(\psi)} = e\psi$ (proof of Claim 2). Now, as A is abelian, all the irreducible characters of A have degree 1 and for each $x \in A$, $\psi(x)$ is an $o(x)$-th root of unity. Hence $\forall\, x \in A$ we have

$$|\eta(x)| = |\eta\downarrow_A^{\mathcal{I}_G(\psi)}(x)| = |e\psi(x)| = e|\psi(x)| = e\cdot 1 = e = \eta(1) \quad\Rightarrow\quad A \subseteq Z(\eta)\,.$$

Therefore, by Remark 17.5, we have

$$\eta(1) \,\Big|\, |\mathcal{I}_G(\psi):Z(\eta)| \,\Big|\, |\mathcal{I}_G(\psi):A|$$

and since $\chi = \eta\uparrow_{\mathcal{I}_G(\psi)}^G$ it follows that

$$\chi(1) = |G:\mathcal{I}_G(\psi)|\eta(1) \,\Big|\, |G:\mathcal{I}_G(\psi)|\cdot|\mathcal{I}_G(\psi):A| = |G:A|\,.$$

■

21 The Theorem of Gallagher

In the context of Clifford theory (Theorem 20.5) we understand that irreducibility of characters is preserved by induction from $\mathcal{I}_G(\psi)$ to G. Thus we need to understand induction of characters from N to $\mathcal{I}_G(\psi)$, in particular what if $G = \mathcal{I}_G(\psi)$. What can be said about $\mathrm{Irr}(G \mid \psi)$?

Lemma 21.1

Let $N \trianglelefteq G$ and let $\psi \in \mathrm{Irr}(N)$ such that $\mathcal{I}_G(\psi) = G$. Then

$$\psi\uparrow_N^G = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi \chi$$

where $e_\chi := \langle \chi\downarrow_N^G, \psi\rangle_N$ is the ramification index of χ in N; in particular

$$\sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2 = |G:N|\,.$$

Proof: Write $\psi\uparrow_N^G = \sum_{\chi\in\mathrm{Irr}(G)} a_\chi \chi$ with suitable $a_\chi = \langle \chi, \psi\uparrow_N^G\rangle_G$. By Frobenius reciprocity, $a_\chi \neq 0$ if and only if $\chi \in \mathrm{Irr}(G \mid \psi)$. But by Theorem 20.5: if $\chi \in \mathrm{Irr}(G|\psi)$, then $\chi\downarrow_N^G = e_\chi \psi$, so that

$$e_\chi = \langle \chi\downarrow_N^G, \psi\rangle_N = \langle \chi, \psi\uparrow_N^G\rangle_G = a_\chi\,.$$

Therefore,

$$|G:N|\psi(1) = \psi\uparrow_N^G(1) = \sum_{\chi\in\mathrm{Irr}(G)} a_\chi \chi(1) = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi \chi(1) = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2 \psi(1) = \psi(1) \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2$$

and it follows that $|G:N| = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2$. ■

Therefore the multiplicities $\{e_\chi\}_{\chi\in\mathrm{Irr}(G)}$ behave like the irreducible character degrees of the factor group G/N. This is not a coincidence in many cases.

Definition 21.2 (*Extension of a character*)

Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$ such that $\psi := \chi\downarrow_N^G$ is irreducible. Then we say that ψ **extends to** G, and χ is an **extension of** ψ.

Exercise 21.3 (*Exercise 28, Sheet 7*)

Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Prove that

$$\chi\downarrow_N^G\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{\mathrm{reg}}) \cdot \chi\,,$$

where χ_{reg} is the regular character of G/N.

Theorem 21.4 (Gallagher)

Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ such that $\psi := \chi\downarrow_N^G \in \mathrm{Irr}(N)$. Then

$$\psi\uparrow_N^G = \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\, \mathrm{Inf}_{G/N}^G(\lambda) \cdot \chi,$$

where the characters $\{\mathrm{Inf}_{G/N}^G(\lambda) \cdot \chi \mid \lambda \in \mathrm{Irr}(G/N)\}$ of G are pairwise distinct and irreducible.

Proof: By Exercise 21.3 we have $\psi\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{\mathrm{reg}}) \cdot \chi$, where χ_{reg} denotes the regular character of G/N. Recall that by Theorem 10.3, $\chi_{\mathrm{reg}} = \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\,\lambda$, so that we have

$$\psi\uparrow_N^G = \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\, \mathrm{Inf}_{G/N}^G(\lambda) \cdot \chi\,.$$

Now, by Lemma 21.1, we have

$$|G:N| = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2 = \langle \psi\uparrow_N^G, \psi\uparrow_N^G\rangle_G = \sum_{\lambda,\mu\in\mathrm{Irr}(G/N)} \lambda(1)\mu(1)\langle \mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi, \mathrm{Inf}_{G/N}^G(\mu)\cdot\chi\rangle_G$$
$$\geqslant \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)^2 = |G:N|\,.$$

Hence equality holds throughout. This proves that

$$\langle \mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi, \mathrm{Inf}_{G/N}^G(\mu)\cdot\chi\rangle = \delta_{\lambda\mu}.$$

By Erercise 13.4, $\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi$ are characters of G and hence the characters $\{\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi \mid \lambda\in\mathrm{Irr}(G/N)\}$ are irreducible and pairwise distinct, as claimed. ■

Therefore, given $\psi\in\mathrm{Irr}(N)$ which extends to $\chi\in\mathrm{Irr}(G)$, we get $\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi$ ($\lambda\in\mathrm{Irr}(G/N)$) as further irreducible characters.

Example 14

Let $N < G$ with $|G:N| = 2$ ($\Rightarrow N \trianglelefteq G$) and let $\psi\in\mathrm{Irr}(N)$. We saw:

- if $\mathcal{I}_G(\psi) = N$ then $\psi\uparrow_N^G\in\mathrm{Irr}(G)$;
- if $\mathcal{I}_G(\psi) = G$ then ψ extends to some $\chi\in\mathrm{Irr}(G)$ and $\psi^G = \chi + \chi'$ with $\chi'\in\mathrm{Irr}(G)\setminus\{\chi\}$. It follows that $\chi' = \chi\cdot\mathrm{sign}$, where sign is the inflation of the sign character of $G/N \cong \mathfrak{S}_2$ to G.