In this chapter we present important methods to construct / relate characters of a group, given characters of subgroups or overgroups. The main idea is that we would like to be able to use the character tables of groups we know already in order to compute the character tables of subgroups or overgroups of these groups.

Notation: throughout this chapter, unless otherwise specified, we let:

· G denote a finite group, $H \leqslant G$ and $N \trianglelefteq G$.

· $K := \mathbb{C}$ be the field of complex numbers;

· $\mathrm{Irr}(G) := \{\chi_1, \ldots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of G;

· $C_1 = [g_1], \ldots, C_r = [g_r]$ denote the conjugacy classes of G, where $g_1, \ldots, g_r$ is a fixed set of representatives; and

· we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

19 Induction and Restriction

We aim at *inducing* and *restricting* characters from subgroups, resp. overgroups. We start with the operation of induction, which is a subtle operation to construct a class function on G from a given class function on a subgroup $H \leqslant G$. We will focus on characters in a second step.

Definition 19.1 (*Induced class function*)

Let $H \leqslant G$ and $\varphi \in Cl(H)$ be a class function on H. Then the **induction of φ from H to G** is

$$\mathrm{Ind}_H^G(\varphi) =: \varphi\!\uparrow_H^G : \quad \begin{aligned} G &\longrightarrow \mathbb{C} \\ \varphi &\mapsto \varphi\!\uparrow_H^G := \tfrac{1}{|H|} \textstyle\sum_{x \in G} \varphi^\circ(x^{-1}gx) , \end{aligned}$$

where for $y \in G$, $\varphi^\circ(y) := \begin{cases} \varphi(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases}$

Remark 19.2

With the notation of Definition 19.1 the function $\varphi\uparrow_H^G$ is again a class function, because for every $g, y \in G$,

$$\varphi\uparrow_H^G (y^{-1}gy) = \frac{1}{|H|} \sum_{x\in G} \varphi^\circ(x^{-1}y^{-1}gyx) \overset{s:=yx}{=} \frac{1}{|H|} \sum_{s\in G} \varphi^\circ(s^{-1}gs) = \varphi\uparrow_H^G (g).$$

Moreover, the map $\mathrm{Ind}_H^G : Cl(H) \longrightarrow Cl(G), \varphi \mapsto \varphi\uparrow_H^G$ is $\mathbb{C}$-linear [Exercise 25(d), Sheet 7].

In contrast, the operation of restriction is based on the more elementary idea that any map can be restricted to a subset of its domain. For class functions / representations / characters we are essentially interested in restricting these (seen as maps) to subgroups.

Definition 19.3 (*Restricted class function*)

Let $H \leqslant G$ and $\psi \in Cl(G)$ be a class function on G. Then the **restriction of ψ from G to H** is

$$\mathrm{Res}_H^G(\psi) = \psi\downarrow_H^G = \psi|_H .$$

This is obviously again a class function on H.

Remark 19.4

If ψ is a character of G afforded by the $\mathbb{C}$-representation $\rho : G \longrightarrow GL(V)$, then clearly $\psi\downarrow_H^G$ is the character afforded by the $\mathbb{C}$-representation $\rho|_H : H \longrightarrow GL(V)$.

Exercise 19.5 (*Exercise 25, Sheet 7*)

Let $H \leqslant J \leqslant G$. Prove that:

(a) $\varphi \in Cl(H) \implies (\varphi\uparrow_H^J)\uparrow_J^G = \varphi\uparrow_H^G$ (transitivity of induction);

(b) $\psi \in Cl(G) \implies (\psi\downarrow_J^G)\downarrow_H^J = \psi\downarrow_H^G$ (transitivity of restriction);

(c) $\varphi \in Cl(H)$ and $\psi \in Cl(G) \implies \psi \cdot \varphi\uparrow_H^G = (\psi\downarrow_H^G \cdot\varphi)\uparrow_H^G$ (Frobenius formula).

Theorem 19.6 (*Frobenius reciprocity*)

Let $H \leqslant G$, let $\varphi \in Cl(H)$ be a class function on H, and let $\psi \in Cl(G)$ be a class function on G. Then

$$\langle\varphi\uparrow_H^G, \psi\rangle_G = \langle\varphi, \psi\downarrow_H^G\rangle_H.$$

Proof: By the definitions of the scalar product and of the induction a direct computation yields:

$$\langle\varphi\uparrow_H^G, \psi\rangle_G = \frac{1}{|G|} \sum_{g\in G} \varphi\uparrow_H^G (g)\psi(g^{-1}) = \frac{1}{|G|} \sum_{g\in G} \frac{1}{|H|} \sum_{x\in G} \varphi^\circ(x^{-1}gx)\psi(g^{-1})$$

$$= \frac{1}{|G||H|} \sum_{s\in G} \sum_{x\in G} \varphi^\circ(s)\psi(xs^{-1}x^{-1})$$

$$= \frac{1}{|H|} \sum_{s\in G} \varphi^\circ(s)\psi(s^{-1}) = \frac{1}{|H|} \sum_{s\in H} \varphi(s^{-1})\psi(s) = \langle\varphi,\downarrow_H^G\rangle_H ,$$

where for the third equality we set $s := x^{-1}gx$ and the fourth equality comes from the fact that ψ is a class function on G. ∎

Corollary 19.7

Let $H \leqslant G$ and let χ be a character of H of degree n. Then the induced class function $\chi \uparrow_H^G$ is a character of G of degree $n|G : H|$.

Proof: For $\psi \in \mathrm{Irr}(G)$ by Frobenius reciprocity we can set

$$m_\psi := \langle \chi \uparrow_H^G, \psi \rangle_G = \langle \chi, \psi \downarrow_H^G \rangle_H \in \mathbb{Z}_{\geqslant 0} \,,$$

which is an integer because both χ and $\psi \downarrow_H^G$ are characters of H. Therefore,

$$\chi \uparrow_H^G = \sum_{\psi \in \mathrm{Irr}(G)} m_\psi \psi$$

is a non-negative integral linear combination of irreducible characters of G, hence a character of G. Moreover,

$$\chi \uparrow_H^G (1) = \frac{1}{|H|} \sum_{x \in G} \chi^\circ(1) = \frac{1}{|H|}|G|\chi(1) = \chi(1)|G : H| \,.$$

$\blacksquare$

Example 11

(a) The restriction of the trivial character of G from G to H is obviously the trivial character of H.

(b) If $H = \{1\}$, then $\mathbf{1}_{\{1\}} \uparrow_{\{1\}}^G = \chi_{\mathrm{reg}}$. Indeed, if $g \in G$ then

$$\mathbf{1}_{\{1\}} \uparrow_{\{1\}}^G (g) = \frac{1}{|\{1\}|} \sum_{x \in G} \underbrace{\mathbf{1}_{\{1\}}^\circ(x^{-1}gx)}_{=0 \text{ unless } g=1} = \delta_{1g}|G| = \chi_{\mathrm{reg}}(g)$$

by Corollary 10.2.

(c) Let $G = S_3$, $H = \langle (1\ 2) \rangle$, and let $\varphi : H \to \mathbb{C}$ with $\varphi(\mathrm{Id}) = 1$, $\varphi((1\ 2)) = -1$ be the sign homomorphism on H. By the remark, it is enough to compute $\varphi \uparrow_H^G$ on representatives of the conjugacy classes of S_3, e.g. Id, $(1\ 2)$ and $(1\ 2\ 3)$:

$$\varphi \uparrow_H^G (\mathrm{Id}) = \frac{1}{2} \sum_{x \in S_3} \varphi^\circ(\mathrm{Id}) = 3 \,,$$

$$\varphi \uparrow_H^G ((1\ 2\ 3)) = \frac{1}{2} \sum_{x \in S_3} \varphi^\circ(x^{-1}(1\ 2\ 3)x) = 0 \,,$$

$$\varphi \uparrow_H^G ((1\ 2)) = \frac{1}{2} \sum_{x \in S_3} \varphi^\circ(x^{-1}(1\ 2)x) = \varphi^\circ((1\ 2)) + \varphi^\circ((1\ 3)) + \varphi^\circ((2\ 3)) = -1 + 0 + 0 = -1 \,.$$

Moreover we see from the character table of S_3 (Example 5) that $\varphi \uparrow_H^G = \chi_2 + \chi_3$. But we can also compute with Frobenius reciprocity, that

$$0 = \langle \varphi, \chi_1 \downarrow_H^G \rangle_H = \langle \varphi \uparrow_H^G, \chi_1 \rangle_G$$

and similarly

$$1 = \langle \varphi, \chi_2 \downarrow_H^G \rangle_H = \langle \varphi \uparrow_H^G, \chi_2 \rangle_G \quad \text{and} \quad 1 = \langle \varphi, \chi_3 \downarrow_H^G \rangle_H = \langle \varphi \uparrow_H^G, \chi_3 \rangle_G \,.$$

Example 12 (*The character table of the alternating group A_5*)

The conjugacy classes of $G = A_5$ are

$$C_1 = \{\text{Id}\} \ , \ C_2 = [(1\ 2)(3\ 4)] \ , \ C_3 = [(1\ 2\ 3)] \ , \ C_4 \cup C_5 = \{5\text{-cycles}\} \ ,$$

i.e. $g_1 = \text{Id}, g_2 = (1\ 2)(3\ 4), g_3 = (1\ 2\ 3)$ and $g \in C_4 \Rightarrow o(g) = 5$ and $g^{-1} \in C_4$ but $g^2, g^3 \in C_5$ so that we can choose $g_4 := (1\ 2\ 3\ 4\ 5)$ and $g_5 := (1\ 3\ 5\ 2\ 4)$. This yields:

$$|\,\text{Irr}(\mathfrak{A}_5)| = 5 \text{ and } |C_1| = 1, |C_2| = 15, |C_3| = 20, |C_4| = |C_5| = 12\,.$$

We obtain the character table of A_5 as follows:

- We know that the trivial character $\mathbf{1}_G = \chi_1$ is one of the irreducible characters, hence we need to determine $\text{Irr}(A_5)\backslash\{\mathbf{1}_G\} = \{\chi_2, \chi_3, \chi_4, \chi_5\}$.

- Now, $H := A_4 \leqslant A_5$ and we have already computed the character table of A_4 in Exercise 14.10 (cf. solution of Exercise 19, Sheet 5). Therefore, inducing the trivial character of A_4 from A_4 to A_5 we obtain that

$$\mathbf{1}_H{\uparrow}_H^G (\text{Id}) = 1 \cdot |G : H| = 5 \quad (\text{see Cor. 19.7})$$
$$\mathbf{1}_H{\uparrow}_H^G ((1\ 2)(3\ 4)) = \tfrac{1}{12} \cdot 12 = 1$$
$$\mathbf{1}_H{\uparrow}_H^G ((1\ 2\ 3)) = \tfrac{1}{12} \cdot 24 = 2$$
$$\mathbf{1}_H{\uparrow}_H^G (5\text{-cycle}) = \tfrac{1}{12} \cdot 0 = 0$$

Now by Frobenius reciprocity

$$\langle \mathbf{1}_H{\uparrow}_H^G, \chi_1 \rangle_G = \langle \mathbf{1}_H, \underbrace{\chi_1{\downarrow}_H^G}_{=\mathbf{1}_H} \rangle_H = 1\,.$$

It follows (check it) that $\langle \mathbf{1}_H{\uparrow}_H^G - \chi_1, \mathbf{1}_H{\uparrow}_H^G - \chi_1 \rangle_G = 1$, so $\mathbf{1}_H{\uparrow}_H^G - \chi_1$ is an irreducible character, say $\chi_4 := \mathbf{1}_H{\uparrow}_H^G - \chi_1$. The values of χ_4 are given by $(4, 0, 1, -1, -1)$ on C_1, C_2, C_3, C_4, C_5 respectively.

- Next, as A_5 is a nonabelian simple group $A_5/[A_5, A_5] = 1$, hence the trivial character is the unique linear character of A_5 and $\chi_2(1), \chi_3(1), \chi_5(1) \geqslant 3$. (You have also proved in Exercise 20, Sheet 5 that simple groups do not have irreducible characters of degree 2.) Then the degree formula yields

$$\chi_2(1)^2 + \chi_3(1)^2 + \chi_5(1)^2 = |A_5| - \chi_1(1)^2 - \chi_4(1)^2 = 20 - 1 - 16 = 43\,.$$

As degrees of characters must divide the group order, it follows from this formula that $\chi_2(1), \chi_3(1), \chi_5(1) \in \{3, 4, 5, 6\}$, but then also that it is not possible to have an irreducible character of degree 6. From this we easily see that only possibility, up to relabelling, is $\chi_2(1) = \chi_3(1) = 3$ and $\chi_5(1) = 5$. Hence at this stage, we already have the following part of the character table:

	C_1	C_2	C_3	C_4	C_5
$|C_k|$	1	15	20	12	12
$|C_G(g_k)|$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3	.	.	.	.
χ_3	3	.	.	.	.
χ_4	4	0	1	-1	-1
χ_5	5	.	.	.	.

· Next, we have that

$$\gcd(\chi_2(1), |C_3|) = \gcd(\chi_3(1), |C_3|) = \gcd(\chi_5(1), |C_4|)$$
$$= \gcd(\chi_5(1), |C_4|) = \gcd(\chi_5(1), |C_5|) = 1,$$

so that the corresponding character values must all be zero by Corollary 17.7 and we get:

	C_1	C_2	C_3	C_4	C_5
$|C_k|$	1	15	20	12	12
$|C_G(g_k)|$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3	.	0	.	.
χ_3	3	.	0	.	.
χ_4	4	0	1	-1	-1
χ_5	5	.	.	0	0

· Applying the Orthogonality Relations yields:
1st, 3rd column $\Rightarrow \chi_5(g_3) = -1$ and the scalar product $\langle \chi_1, \chi_5 \rangle_G = 0 \Rightarrow \chi_5(g_2) = 1$.

· Finally, to fill out the remaining gaps, we can induce from the cyclic subgroup $Z_5 := \langle (1\,2\,3\,4\,5) \rangle \leqslant A_5$: choosing the non-trivial irreducible character ψ of Z_5 which was denoted "χ_3" in Example 4 gives

$$\psi \uparrow_{Z_5}^G = (12, 0, 0, \zeta^2 + \zeta^3, \zeta + \zeta^4)$$

where $\zeta = \exp(2\pi i/5)$ is a primitive 5-th root of unity. Then we compute that

$$\langle \psi \uparrow_{Z_5}^G, \chi_4 \rangle_G = 1 = \langle \psi \uparrow_{Z_5}^G, \chi_5 \rangle_G \quad \Longrightarrow \quad \psi \uparrow_{Z_5}^G - \chi_4 - \chi_5 = (3, -1, 0, -\zeta - \zeta^4, -\zeta^2 - \zeta^3)$$

and this character must be irreducible, because it is not the sum of 3 copies of the trivial character. Hence we set $\chi_2 := \psi \uparrow_{Z_5}^G - \chi_4 - \chi_5$ and the values of χ_3 then easily follow from the 2nd Othogonality Relations:

	C_1	C_2	C_3	C_4	C_5
$|C_k|$	1	15	20	12	12
$|C_G(g_k)|$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3	-1	0	$-\zeta - \zeta^4$	$-\zeta^2 - \zeta^3$
χ_3	3	-1	0	$-\zeta^2 - \zeta^3$	$-\zeta - \zeta^4$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

Remark 19.8 (*Induction of KH-modules*)

If you have attended the lecture *Commutative Algebra* you have studied the *tensor product of modules*. In the M.Sc. lecture *Representation Theory* you will see that induction of modules is defined through a tensor product, extending the scalars from $\mathbb{C}H$ to $\mathbb{C}G$. More precisely, if M is a KH-module, then the induction of M from H to G is defined to be $KG \otimes_{KH} M$. Moreover, if M affords the character χ, then $KG \otimes_{KH} M$ affords the character $\chi \uparrow_H^G$.