13 Tensor Products of Representations and Characters

Tensor products of vector spaces and matrices are recalled/introduced in Appendix C. We are now going to use this construction to build *products* of characters.

Proposition 13.1

Let G and H be finite groups, and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : H \longrightarrow \mathrm{GL}(W)$ be $\mathbb{C}$-representations with characters χ_V and χ_W respectively. Then

$$\begin{array}{rccl} \rho_V \otimes \rho_W : & G \times H & \longrightarrow & \mathrm{GL}(V \otimes_{\mathbb{C}} W) \\ & (g,h) & \mapsto & (\rho_V \otimes \rho_W)(g,h) := \rho_V(g) \otimes \rho_W(h) \end{array}$$

(where $\rho_V(g) \otimes \rho_W(h)$ is the tensor product of the $\mathbb{C}$-endomorphisms $\rho_V(g) : V \longrightarrow V$ and $\rho_W(h) : W \longrightarrow W$ as defined in Lemma-Definition C.4) is a $\mathbb{C}$-representation of $G \times H$, called the **tensor product** of ρ_V and ρ_W, and the corresponding character, which we denote by $\chi_{V \otimes_{\mathbb{C}} W}$, is

$$\chi_{V \otimes_{\mathbb{C}} W} = \chi_V \cdot \chi_W \,,$$

where $\chi_V \cdot \chi_W(g,h) := \chi_V(g) \cdot \chi_W(h) \ \forall \ (g,h) \in G \times H$.

Proof: First note that $\rho_V \otimes \rho_W$ is well-defined by Lemma-Definition C.4 and it is a group homomorphism because

$$\begin{aligned} (\rho_V \otimes \rho_W)(g_1g_2, h_1h_2)[v \otimes w] &= \big(\rho_V(g_1g_2) \otimes \rho_W(h_1h_2)\big)[v \otimes w] \\ &= \rho_V(g_1g_2)[v] \otimes \rho_W(h_1h_2)[w] \\ &= \rho_V(g_1) \circ \rho_V(g_2)[v] \otimes \rho_W(h_1) \circ \rho_W(h_2)[w] \\ &= \rho_V(g_1) \otimes \rho_W(h_1)\big[\rho_V(g_2)[v] \otimes \rho_W(h_2)[w]\big] \\ &= \big(\rho_V(g_1) \otimes \rho_W(h_1)\big) \circ \big(\rho_V(g_2) \otimes \rho_W(h_2)\big)[v \otimes w] \\ &= \big(\rho_V \otimes \rho_W\big)(g_1, h_1) \circ \big(\rho_V \otimes \rho_W\big)(g_2, h_2)[v \otimes w] \end{aligned}$$

$\forall \ g_1, g_2 \in G,\ h_1, h_2 \in H,\ v \in V,\ w \in W.$ Furthermore, for each $g \in G$ and each $h \in H$,

$$\chi_{V \otimes_{\mathbb{C}} W}(g,h) = \mathrm{Tr}\left((\rho_V \otimes \rho_W)(g,h)\right) = \mathrm{Tr}\left(\rho_V(g) \otimes \rho_W(h)\right) = \mathrm{Tr}\left(\rho_V(g)\right) \cdot \mathrm{Tr}\left(\rho_W(h)\right) = \chi_V(g) \cdot \chi_W(h)$$

by Lemma-Definition C.4, hence $\chi_{V \otimes_{\mathbb{C}} W} = \chi_V \cdot \chi_W$. ■

Remark 13.2

The diagonal inclusion $\iota : G \longrightarrow G \times G, g \mapsto (g,g)$ of G in the product $G \times G$ is a group homomorphism with $\iota(G) \cong G$. Therefore, if $G = H$, then

$$G \xrightarrow{\iota} G \times G \xrightarrow{\chi_V \cdot \chi_W} \mathbb{C}, g \mapsto (g,g) \mapsto \chi_V(g) \cdot \chi_W(g)$$

becomes a character of G, which we also denote by $\chi_V \cdot \chi_W$.

Corollary 13.3

If G and H are finite groups, then $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

Proof: [Exercise 15(c), Sheet 4]. Hint: Use Corollary 9.8(d) and the degree formula. ■

Exercise 13.4 (*Exercise 15(a)+(b), Sheet 4*)

(a) If $\lambda, \chi \in \mathrm{Irr}(G)$ and $\lambda(1) = 1$, then $\lambda \cdot \chi \in \mathrm{Irr}(G)$.

(b) The set $\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ of linear characters of a finite group G forms a group for the product of characters.

14 Normal Subgroups and Inflation

Whenever a group homomorphism $G \longrightarrow H$ and a representation of H are given, we obtain a representation of G by composition. In particular, we want to apply this principle to normal subgroups $N \trianglelefteq G$ and the corresponding quotient homomorphism, which we always denote by $\pi : G \longrightarrow G/N, g \mapsto gN$.

We will see that by this means, copies of the character tables of quotient groups of G all appear in the character table of G. This observation, although straightforward, will allow us to fill out the character table of a group very rapidly, provided it possesses normal subgroups.

Definition 14.1 (*Inflation*)

Let $N \trianglelefteq G$ and let $\pi : G \longrightarrow G/N, g \mapsto gN$ be the quotient homomorphism. Given a $\mathbb{C}$-representation $\rho : G/N \longrightarrow \mathrm{GL}(V)$, we set

$$\mathrm{Inf}_{G/N}^{G}(\rho) := \rho \circ \pi : G \longrightarrow \mathrm{GL}(V)\,.$$

This is a $\mathbb{C}$-representation of G, called the **inflation of ρ from G/N to G**. If the character of ρ is χ, then we denote by $\mathrm{Inf}_{G/N}^{G}(\chi)$ the character of $\mathrm{Inf}_{G/N}^{G}(\rho)$ and call it the **inflation of χ from G/N to G**.

Note that some texts also call $\mathrm{Inf}_{G/N}^{G}(\rho)$ (resp. $\mathrm{Inf}_{G/N}^{G}(\chi)$) the *lift* of ρ (resp. χ) along π.

Remark 14.2

The values of the character $\mathrm{Inf}_{G/N}^{G}(\chi)$ of G are obtained from those of χ as follows. If $g \in G$, then

$$\mathrm{Inf}_{G/N}^{G}(\chi)(g) = \mathrm{Tr}\left((\rho \circ \pi)(g)\right) = \mathrm{Tr}\left(\rho(gN)\right) = \chi(gN)\,.$$

Exercise 14.3 (*Exercise 16, Sheet 4*)

Let $N \trianglelefteq G$ and let $\rho : G/N \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G/N with character χ.

(a) Prove that if ρ is irreducible, then so is $\mathrm{Inf}_{G/N}^{G}(\rho)$.

(b) Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$ provided that ρ is faithful.

Definition 14.4 (*Kernel of a character*)

The **kernel of a character** χ of G is $\ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$.

Example 6

(a) $\chi = \mathbf{1}_G$ the trivial character $\Rightarrow \ker(\chi) = G$.

(b) $G = \mathfrak{S}_3$, $\chi = \chi_2$ the sign character $\Rightarrow \ker(\chi) = C_1 \cup C_3 = \langle(123)\rangle$; whereas $\ker(\chi_3) = \{1\}$. (See Example 5.)

Lemma 14.5

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G with character ψ. Then $\ker(\psi) = \ker(\rho)$, thus is a normal subgroup of G.

Proof: [Exercise 17(a), Sheet 5] ■

Theorem 14.6

Let $N \trianglelefteq G$. Then

$$\begin{array}{rccl} \mathrm{Inf}^G_{G/N} : & \{\text{characters of } G/N\} & \longrightarrow & \{\text{characters } \psi \text{ of } G \mid N \leqslant \ker(\psi)\} \\ & \chi & \mapsto & \mathrm{Inf}^G_{G/N}(\chi) \end{array}$$

is a bijection and so is its restriction to the irreducible characters

$$\begin{array}{rccl} \mathrm{Inf}^G_{G/N} : & \mathrm{Irr}(G/N) & \longrightarrow & \{\psi \in \mathrm{Irr}(G) \mid N \leqslant \ker(\psi)\} \\ & \chi & \mapsto & \mathrm{Inf}^G_{G/N}(\chi)\,. \end{array}$$

Proof: First we prove that the first map is well-defined and bijective.

- Let χ be a character of G/N afforded by the $\mathbb{C}$-representation $\rho : G/N \longrightarrow \mathrm{GL}(V)$. By Remark 14.2, N is in the kernel of $\mathrm{Inf}^G_{G/N}(\chi)$, hence the first map is well-defined.
- Now let ψ be a character of G with $N \leqslant \ker(\psi)$ and assume ψ is afforded by the $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$.

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & \mathrm{GL}(V) \\ {\scriptstyle\pi}\downarrow & \circlearrowleft & \nearrow {\scriptstyle\exists!\,\widetilde{\rho}} \\ G/N & & \end{array}$$

By Lemma 14.5 we have $\ker(\psi) = \ker(\rho) \geqslant N$. Therefore, by the universal property of the quotient, ρ induces a unique $\mathbb{C}$-representation $\widetilde{\rho} : G/N \longrightarrow \mathrm{GL}(V)$ with the property that $\widetilde{\rho} \circ \pi = \rho$.

It follows that $\rho = \mathrm{Inf}^G_{G/N}(\widetilde{\rho})$ and $\psi = \mathrm{Inf}^G_{G/N}(\chi)$. Thus the 1st map is surjective. Its injectivity is clear.

The second map is well-defined by the above and Exercise 14.3(a). It is injective because it is just the restriction of the 1st map to the $\mathrm{Irr}(G/N)$, whereas it is surjective by the same argument as above as the constructed representation $\widetilde{\rho}$ is clearly irreducible if ρ is because $\widetilde{\rho} \circ \pi = \rho$. ■

Exercise 14.7 (*Exercise 17(b), Sheet 5*)

Let G be a finite group. Prove that if $N \trianglelefteq G$, then

$$N = \bigcap_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq \ker(\chi)}} \ker(\chi)\,.$$

It follows immediately from the above exercise that the lattice of normal subgroups of G can be read off from its character table. The theorem also implies that it can be read off from the character table, whether the group is abelian or simple.

Corollary 14.8

(a) Inflation from the abelianisation induces a bijection

$$\mathrm{Inf}_{G/G'}^{G} : \mathrm{Irr}(G/G') \xrightarrow{\sim} \{\psi \in \mathrm{Irr}(G) \mid \psi(1) = 1\} ;$$

in particular, G has precisely $|G : G'|$ linear characters.

(b) The group G is abelian if and only if all its irreducible characters are linear.

Proof: (a) First, we claim that if $\psi \in \mathrm{Irr}(G)$ is linear, then G' is in its kernel. Indeed, if $\psi(1) = 1$, then $\psi : G \longrightarrow \mathbb{C}^\times$ is a group homomorphism. Therefore, as $\mathbb{C}^\times$ is abelian,

$$\psi([g,h]) = \psi(ghg^{-1}h^{-1}) = \psi(g)\psi(h)\psi(g)^{-1}\psi(h)^{-1} = \psi(g)\psi(g)^{-1}\psi(h)\psi(h)^{-1} = 1$$

for all $g, h \in G$, and hence $G' = \langle [g,h] \mid g, h \in G\rangle \leqslant \ker(\chi)$. In addition, any irreducible character of G/G' is linear by Proposition 6.1 because G/G' is abelian. Thus Theorem 14.6 yields a bijection

$$\mathrm{Irr}(G/G') \xrightarrow[\mathrm{Inf}_{G/G'}^{G}]{\sim} \{\psi \in \mathrm{Irr}(G) \mid G' \leqslant \ker(\psi)\} = \{\psi \in \mathrm{Irr}(G) \mid \psi(1) = 1\},$$

as required.

(b) If G is abelian, then $G/G' = G$. Hence the claim follows from (a). ■

Corollary 14.9

A finite group G is simple $\iff \chi(g) \neq \chi(1)\ \ \forall\, g \in G\backslash\{1\}$ and $\forall\, \chi \in \mathrm{Irr}(G)\backslash\{\mathbf{1}_G\}$.

Proof: [Exercise 18, Sheet 5] ■