Chapter 4. The Character Table

In Chapter 3 we have proved that for any finite group G the equality $|\operatorname{Irr}(G)| = |C(G)| =: r$ holds. Thus the values of the irreducible characters of G can be recorded in an $r \times r$-matrix, called the *character table* of G. The entries of this matrix are related to each other in subtle manners, many of which are encapsulated in the 1st Orthogonality Relations and their consequences, as for example the degree formula. Our aim in this chapter is to develop further tools and methods to compute character tables.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- $K := \mathbb{C}$ be the field of complex numbers;
- $\operatorname{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of G;
- $C_1 = [g_1], \ldots, C_r = [g_r]$ denote the conjugacy classes of G, where $g_1, \ldots, g_r$ is a fixed set of representatives; and
- we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

11 The Character Table of a Finite Group

Definition 11.1 (*Character table*)

The **character table of** G is the matrix $X(G) := \left(\chi_i(g_j)\right)_{ij} \in M_r(\mathbb{C})$.

Example 4 (*The character table of a cyclic group*)

Let $G = \langle g \mid g^n = 1\rangle$ be cyclic of order $n \in \mathbb{Z}_{>0}$. Since G is abelian,

$$\operatorname{Irr}(G) = \{\text{linear characters of } G\}$$

by Proposition 6.1 and $|\operatorname{Irr}(G)| = |G| = n$. Moreover, each conjugacy class is a singleton:

$$\forall\, 1 \leqslant j \leqslant r: \quad C_j = \{g_j\} \text{ and we set } g_j := g^{j-1}.$$

Let ζ be a primitive n-th root of unity in $\mathbb{C}$, so that $\{\zeta^i \mid 1 \leqslant i \leqslant n\}$ are all the n-th roots of unity. Now, each $\chi_i : G \to \mathbb{C}^\times$ is a group homomorphism and is determined by $\chi_i(g)$, which has to be an n-th root of $1_\mathbb{C}$. Therefore, we have n possibilities for $\chi_i(g)$. We set

$$\chi_i(g) := \zeta^{i-1} \quad \forall\, 1 \leqslant i \leqslant n \quad \Rightarrow \quad \chi_i(g^j) = \zeta^{(i-1)j} \quad \forall\, 1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-1$$

Thus the character table of G is

$$X(G) = \Big(\chi_i(g_j)\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} = \Big(\chi_i(g^{j-1})\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} = \Big(\zeta^{(i-1)(j-1)}\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}},$$

which we visualise as follows:

	1	g	g^2	$\dots$	g^{n-1}
$\chi_1 = 1_G$	1	1	1	$\dots$	1
χ_2	1	ζ	ζ^2	$\dots$	ζ^{n-1}
χ_3	1	ζ^2	ζ^4	$\dots$	$\zeta^{2(n-1)}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
χ_n	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\dots$	$\zeta^{(n-1)^2}$

Example 5 (*The character table of S_3*)

Let now $G := S_3$ be the symmetric group on 3 letters. Recall from the AGS/*Einführung in die Algebra* that the conjugacy classes of S_3 are

$$C_1 = \{\mathrm{Id}\},\ C_2 = \{(1\ 2), (1\ 3), (2\ 3)\},\ C_3 = \{(1\ 2\ 3), (1\ 3\ 2)\}$$

$$\Rightarrow \quad r = 3, |C_1| = 1, |C_2| = 3, |C_3| = 2\,.$$

In Example 2(d) we have exhibited three non-equivalent irreducible matrix representations of S_3, which we denoted ρ_1, ρ_2, ρ_3. For each $1 \leqslant i \leqslant 3$ let χ_i be the character of ρ_i and n_i be its degree, so that $n_1 = n_2 = 1$ and $n_3 = 2$. Hence

$$n_1^2 + n_2^2 + n_3^2 = 6 = |G|\,.$$

Therefore, the degree formula tells us that ρ_1, ρ_2, ρ_3 are all the irreducible matrix representations of S_3, up to equivalence. We note that $n_1 = n_2 = 1$, $n_3 = 2$ is in fact the unique solution (up to relabelling) to the equation given by the degree formula! Taking traces of the matrices in Example 2(d) yields the character table of S_3.

	Id	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

In the next sections we want to develop further techniques to compute character tables of finite groups, before we come back to further examples of such tables for larger groups.

Exercise 11.2 (*Exercise 13(c), Sheet 4*)

Compute the character table of the Klein-four group $C_2 \times C_2$.

12 The 2nd Orthogonality Relations

The 1st Orthogonality Relations provide us with orthogonality relations between the rows of the character table. They can be rewritten as follows in terms of matrices.

Exercise 12.1 (*Exercise 14, Sheet 4*)

Let G be a finite group. Set $X := X(G)$ and

$$C := \begin{bmatrix} |C_G(g_1)| & 0 & \cdots & 0 \\ 0 & |C_G(g_2)| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & |C_G(g_r)| \end{bmatrix} \in M_r(\mathbb{C})\,.$$

Use the Orbit-Stabiliser Theorem in order to prove that the 1st Orthogonality Relations can be rewritten under the form

$$XC^{-1}\overline{X}^{\mathrm{Tr}} = I_r\,,$$

where $\overline{X}^{\mathrm{Tr}}$ denotes the transpose of the complex-conjugate $\overline{X}$ of the character table X of G. Deduce that the character table is invertible.

There are also some orthogonality relations between the columns of the character table. These can easily be deduced from the 1st Orthogonality Relations given above in terms of matrices.

Theorem 12.2 (*2nd Orthogonality Relations*)

With the notation of Exercise 12.1 we have

$$X^{\mathrm{Tr}}\,\overline{X} = C\,.$$

In other words,

$$\sum_{\chi\in\mathrm{Irr}(G)} \chi(g_i)\overline{\chi(g_j)} = \delta_{ij}\frac{|G|}{|[g_i]|} = \delta_{ij}|C_G(g_i)| \qquad \forall\, 1\leqslant i,j\leqslant r\,.$$

Proof: Taking complex conjugation of the formula given by the 1st Orthogonality Relations (Exercise 12.1) yields:

$$XC^{-1}\overline{X}^{\mathrm{Tr}} = I_r \qquad \Longrightarrow \qquad \overline{X}C^{-1}X^{\mathrm{Tr}} = I_r$$

Now, since X is invertible, so are all the matrices in the above equations and hence $X^{\mathrm{Tr}} = \left(\overline{X}C^{-1}\right)^{-1}$. It follows that

$$X^{\mathrm{Tr}}\,\overline{X} = \left(\overline{X}C^{-1}\right)^{-1}\overline{X} = C\overline{X}^{-1}\overline{X} = C\,.$$

The second formula is now obtained by considering the entry (i,j) in the above matrix equation for all $1\leqslant i,j\leqslant r$. ■

Exercise 12.3 (*Exercise 13(a), Sheet 4*)

Prove that the degree formula can be read off from the 2nd Orthogonality Relations.