Chapter 3. Characters of Finite Groups

We now introduce the concept of a *character* of a finite group. These are functions $\chi : G \longrightarrow \mathbb{C}$, obtained from the representations of the group G by taking traces. Characters have many remarkable properties, and they are the fundamental tools for performing computations in representation theory. They encode a lot of information about the group itself and about its representations in a more compact and efficient manner.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- $K := \mathbb{C}$ be the field of complex numbers; and
- V denote a $\mathbb{C}$-vector space such that $\dim_{\mathbb{C}}(V) < \infty$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

7 Characters

Definition 7.1 (*Character, linear character*)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation. The **character** of ρ_V is the function

$$\begin{array}{rccl} \chi_V : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \chi_V(g) := \mathrm{Tr}\left(\rho_V(g)\right) . \end{array}$$

We also say that ρ_V (or the $\mathbb{C}G$-module V) **affords** the character χ_V. If the degree of ρ_V is one, then χ_V is called a **linear** character.

Remark 7.2

(a) Again, we allow ourselves to transport terminology from representations to characters. For example, if ρ_V is irreducible (faithful,...), then the character χ_V is also called **irreducible** (**faithful**,...).
With this terminology, it makes sense to let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G.

(b) Recall that in *linear algebra* (see GDM) the trace of a linear endomorphism φ may be concretely computed by taking the trace of the matrix of φ in a chosen basis of the vector space, and this is independent of the choice of the basis.

Thus to compute characters: choose an ordered basis B of V and obtain $\forall\, g \in G$:

$$\chi_V(g) = \mathrm{Tr}\left(\rho_V(g)\right) = \mathrm{Tr}\left(\left(\rho_V(g)\right)_B\right)$$

(c) For a matrix representation $R : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$, the character of R is then

$$\begin{array}{rccl} \chi_R : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \chi_R(g) := \mathrm{Tr}\left(R(g)\right) . \end{array}$$

Example 3

The character of the trivial representation of G is the function $\mathbf{1}_G : G \longrightarrow \mathbb{C}, g \mapsto 1$ and is called **the trivial character** of G.

Lemma 7.3

Equivalent representations have the same character.

Proof: If $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ are two $\mathbb{C}$-representations, and $\alpha : V \longrightarrow W$ is an isomorphism of representations, then

$$\rho_W(g) = \alpha \circ \rho_V(g) \circ \alpha^{-1} \quad \forall\, g \in G .$$

Now, by the properties of the trace (GDM) for two $\mathbb{C}$-endomorphisms β, γ of V we have $\mathrm{Tr}(\beta \circ \gamma) = \mathrm{Tr}(\gamma \circ \beta)$, hence for every $g \in G$ we have

$$\chi_W(g) = \mathrm{Tr}\left(\rho_W(g)\right) = \mathrm{Tr}\left(\alpha \circ \rho_V(g) \circ \alpha^{-1}\right) = \mathrm{Tr}\big(\rho_V(g) \circ \underbrace{\alpha^{-1} \circ \alpha}_{=\mathrm{Id}_V}\big) = \mathrm{Tr}\left(\rho_V(g)\right) = \chi_V(g) .$$

■

Properties 7.4 (*Elementary properties*)

Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let $g \in G$. Then the following assertions hold:

(a) $\chi_V(1_G) = \dim_{\mathbb{C}} V$;

(b) $\chi_V(g) = \varepsilon_1 + \ldots + \varepsilon_n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are $o(g)$-th roots of unity in $\mathbb{C}$ and $n = \dim_{\mathbb{C}} V$;

(c) $|\chi_V(g)| \leqslant \chi_V(1_G)$;

(d) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$;

(e) if $\rho_V = \rho_{V_1} \oplus \rho_{V_2}$ is the direct sum of two subrepresentations, then $\chi_V = \chi_{V_1} + \chi_{V_2}$.

Proof:

(a) $\rho_V(1_G) = \mathrm{Id}_V$ because representations are group homomorphisms, hence $\chi_V(1_G) = \dim_{\mathbb{C}} V$.

(b) This follows directly from the diagonalisation theorem (Theorem 6.2).

(c) By (b) we have $\chi_V(g) = \varepsilon_1 + \ldots + \varepsilon_n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are roots of unity in $\mathbb{C}$. Hence, applying the triangle inequality repeatedly, we obtain that

$$|\chi_V(g)| = |\varepsilon_1 + \ldots + \varepsilon_n| \leqslant \underbrace{|\varepsilon_1|}_{=1} + \ldots + \underbrace{|\varepsilon_n|}_{=1} = \dim_{\mathbb{C}} V \overset{(a)}{=} \chi_V(1_G) .$$

(d) Again by the diagonalisation theorem, there exists an ordered $\mathbb{C}$-basis B of V and $o(g)$-th roots of unity $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{C}$ such that

$$\left(\rho_V(g)\right)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{bmatrix}.$$

Therefore

$$\left(\rho_V(g^{-1})\right)_B = \begin{bmatrix} \varepsilon_1^{-1} & 0 & \cdots & 0 \\ 0 & \varepsilon_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n^{-1} \end{bmatrix} = \begin{bmatrix} \overline{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & \overline{\varepsilon_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \overline{\varepsilon_n} \end{bmatrix}$$

and it follows that $\chi_V(g^{-1}) = \overline{\varepsilon_1} + \ldots + \overline{\varepsilon_n} = \overline{\varepsilon_1 + \ldots + \varepsilon_n} = \overline{\chi_V(g)}$.

(e) For $i = 1, 2$ let B_i be an ordered $\mathbb{C}$-basis of V_i and consider the $\mathbb{C}$-basis $B := B_1 \sqcup B_2$ of V. Then, by Remark 3.2 for every $g \in G$ we have

$$\left(\rho_V(g)\right)_B = \left[\begin{array}{c|c} \left(\rho_{W_1}(g)\right)_{B_1} & 0 \\ \hline 0 & \left(\rho_{W_2}(g)\right)_{B_2} \end{array}\right],$$

hence $\chi_V(g) = \mathrm{Tr}\left(\rho_V(g)\right) = \mathrm{Tr}\left(\rho_{V_1}(g)\right) + \mathrm{Tr}\left(\rho_{V_2}(g)\right) = \chi_{V_1}(g) + \chi_{V_2}(g)$. ■

Corollary 7.5

Any character is a sum of irreducible characters.

Proof: By Corollary 3.6 to Maschke's theorem, any $\mathbb{C}$-representation can be written as the direct sum of irreducible subrepresentations. Thus the claim follows from Properties 7.4(e). ■

Notation 7.6

Recall from group theory (*Einfürung in die Algebra*) that a group G *acts on itself by conjugation* via

$$\begin{array}{ccl} G \times G & \longrightarrow & G \\ (g, x) & \mapsto & gxg^{-1} =: {}^g x\,. \end{array}$$

The orbits of this action are the *conjugacy classes* of G, we denote them by $[x] := \{ {}^g x \mid g \in G\}$, and we write $C(G) := \{[x] \mid x \in G\}$ for the set of all conjugacy classes of G.

The stabiliser of $x \in G$ is its *centraliser* $C_G(x) = \{g \in G \mid {}^g x = x\}$ and the orbit-stabiliser theorem yields

$$|C_G(x)| = \frac{|G|}{|[x]|}\,.$$

Moreover, a function $f : G \longrightarrow \mathbb{C}$ which is constant on each conjugacy class of G, i.e. such that $f(gxg^{-1}) = f(x)\ \forall\, g, x \in G$, is called a **class function** (on G).

Lemma 7.7

Characters are class functions.

Proof: Let $\rho_V : G \longrightarrow GL(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Again, because by the properties of the trace (GDM) $\text{Tr}(\beta \circ \gamma) = \text{Tr}(\gamma \circ \beta)$ for all $\mathbb{C}$-endomorphisms β, γ of V, it follows that for all $g, x \in G$ we have

$$\begin{aligned}\chi_V(gxg^{-1}) = \text{Tr}\left(\rho_V(gxg^{-1})\right) &= \text{Tr}\left(\rho_V(g)\rho_V(x)\rho_V(g)^{-1}\right)\\ &= \text{Tr}\big(\rho_V(x)\underbrace{\rho_V(g)\rho_V(g)^{-1}}_{=\text{Id}_V}\big) = \text{Tr}\left(\rho_V(x)\right) = \chi_V(x)\,.\end{aligned}$$

■

Exercise 7.8 (*Exercise 9, Sheet 3*)

Let $\rho_V : G \longrightarrow GL(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Prove the following statements.

(a) If $g \in G$ is conjugate to g^{-1}, then $\chi_V(g) \in \mathbb{R}$.

(b) If $g \in G$ is an element of order 2, then $\chi_V(g) \in \mathbb{Z}$ and $\chi_V(g) \equiv \chi_V(1) \pmod 2$.

Exercise 7.9 (The dual representation / the dual character [Exercise 10, Sheet 3])

Let $\rho_V : G \longrightarrow GL(V)$ be a $\mathbb{C}$-representation.

(a) Prove that:

(i) the dual space $V^* := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is endowed with the structure of a $\mathbb{C}G$-module via

$$\begin{array}{ccc} G \times V^* & \longrightarrow & V^* \\ (g, f) & \mapsto & g.f \end{array}$$

where $(g.f)(v) := f(g^{-1}v)\ \forall\, v \in V$;

(ii) the character of the associated $\mathbb{C}$-representation ρ_{V*} is then $\chi_{V*} = \overline{\chi_V}$; and

(iii) if ρ_V decomposes as a direct sum $\rho_{V_1} \oplus \rho_{V_2}$ of two subrepresentations, then $\rho_{V*} = \rho_{V_1^*} \oplus \rho_{V_2^*}$.

(b) Determine the duals of the 3 irreducible representations of S_3 given in Example 2(d).

8 Orthogonality of Characters

We are now going to make use of results from the linear algebra (GDM) on the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on G in order to develop further fundamental properties of characters.

Notation 8.1

We let $\mathcal{F}(G, \mathbb{C}) := \{f : G \longrightarrow \mathbb{C} \mid f \text{ function}\}$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on G. Clearly $\dim_{\mathbb{C}} \mathcal{F}(G, \mathbb{C}) = |G|$ because $\{\delta_g : G \longrightarrow \mathbb{C}, h \mapsto \delta_{gh} \mid g \in G\}$ is a $\mathbb{C}$-basis (see GDM). Set $\mathcal{C}l(G) := \{f \in \mathcal{F}(G, \mathbb{C}) \mid f \text{ is a class function}\}$. This is clearly a $\mathbb{C}$-subspace of $\mathcal{F}(G, \mathbb{C})$, called the **space of class functions on** G.

Exercise 8.2 (*Exercise 11, Sheet 3*)

Find a $\mathbb{C}$-basis of $\mathcal{C}l(G)$ and deduce that $\dim_{\mathbb{C}} \mathcal{C}l(G) = |C(G)|$.

Proposition 8.3

The binary operation

$$\begin{array}{rccc} \langle\, ,\, \rangle_G : & \mathcal{F}(G,\mathbb{C}) \times \mathcal{F}(G,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & (f_1, f_2) & \mapsto & \langle f_1, f_2 \rangle_G := \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)} \end{array}$$

is a scalar product on $\mathcal{F}(G,\mathbb{C})$.

Proof: It is straightforward to check that $\langle\, ,\, \rangle_G$ is sesquilinear and Hermitian (Exercise 11, Sheet 3); it is positive definite because for every $f \in \mathcal{F}(G,\mathbb{C})$,

$$\langle f, f \rangle = \frac{1}{|G|} \sum_{g\in G} f(g)\overline{f(g)} = \frac{1}{|G|} \sum_{g\in G} \underbrace{|f(g)|^2}_{\in \mathbb{R}_{\geqslant 0}} \geqslant 0$$

and moreover $\langle f, f \rangle = 0$ if and only if $f = 0$. ■

Remark 8.4

Obviously, the scalar product $\langle\, ,\, \rangle_G$ restricts to a scalar product on $\mathcal{C}l(G)$. Moreover, if f_2 is a character of G, then by Property 7.4(d) we can write

$$\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)} = \frac{1}{|G|} \sum_{g\in G} f_1(g) f_2(g^{-1})\,.$$