5 Schur's Lemma and Schur's Relations

Schur's Lemma is a basic result concerning simple modules, or in other words irreducible representations. Though elementary to state and prove, it is fundamental to representation theory of finite groups.

Theorem 5.1 (Schur's Lemma)

(a) Let V, W be simple KG-modules. Then the following assertions hold.

  (i) Any homomorphism of KG-modules $\varphi : V \longrightarrow V$ is either zero or invertible. In other words $\text{End}_{KG}(V)$ is a skew-field.

  (ii) If $V \not\cong W$, then $\text{Hom}_{KG}(V, W) = 0$.

(b) If K is an algebraically closed field and V is a simple KG-module, then

$$\text{End}_{KG}(V) = \{\lambda \, \text{Id}_V \mid \lambda \in K\} \cong K \, .$$

Notice that here we state Schur's Lemma in terms of modules, rather than in terms of representations, because part (a) holds in greater generality for arbitrary unital associative rings and part (b) holds for finite-dimensional algebras over an algebraically closed field.

Proof:

(a) First, we claim that every $\varphi \in \text{Hom}_{KG}(V, W) \setminus \{0\}$ admits an inverse in $\text{Hom}_{KG}(W, V)$. Indeed, $\varphi \neq 0 \implies \ker \varphi \subsetneq V$ is a proper KG-submodule of V and $\{0\} \neq \text{Im}\,\varphi$ is a non-zero KG-submodule of W. But then, on the one hand, $\ker \varphi = \{0\}$, because V is simple, hence φ is injective, and on the other hand, $\text{Im}\,\varphi = W$ because W is simple. It follows that φ is also surjective, hence bijective. Therefore, by Properties A.7, φ is invertible with inverse $\varphi^{-1} \in \text{Hom}_{KG}(W, V)$.

Now, (ii) is straightforward from the above. For (i), first recall that $\text{End}_{KG}(V)$ is a ring (see Notation A.8), which is obviously non-zero as $\text{End}_{KG}(V) \ni \text{Id}_V$ and $\text{Id}_V \neq 0$ because $V \neq 0$ since it is simple. Thus, as any $\varphi \in \text{End}_{KG}(V) \setminus \{0\}$ is invertible, $\text{End}_{KG}(V)$ is a skew-field.

(b) Let $\varphi \in \text{End}_{KG}(V)$. Since $K = \overline{K}$, φ has an eigenvalue $\lambda \in K$. Let $v \in V \setminus \{0\}$ be an eigenvector of φ for λ. Then $(\varphi - \lambda \, \text{Id}_V)(v) = 0$. Therefore, $\varphi - \lambda \, \text{Id}_V$ is not invertible and

$$\varphi - \lambda \, \text{Id}_V \in \text{End}_{KG}(V) \overset{(a)}{\implies} \varphi - \lambda \, \text{Id}_V = 0 \implies \varphi = \lambda \, \text{Id}_V \, .$$

Hence $\text{End}_{KG}(V) \subseteq \{\lambda \, \text{Id}_V \mid \lambda \in K\}$, but the reverse inclusion also obviously holds, so that

$$\text{End}_{KG}(V) = \{\lambda \, \text{Id}_V\} \cong K \, .$$

■

Exercise 5.2 (*Exercise 8, Sheet 2*)

Prove that in terms of matrix representations the following statement holds:

Lemma 5.3 (*Schur's Lemma for matrix representations*)

Let $R : G \longrightarrow \text{GL}_n(K)$ and $R' : G \longrightarrow \text{GL}_{n'}(K)$ be two irreducible matrix representations. If there exists $A \in M_{n \times n'}(K) \setminus \{0\}$ such that $AR'(g) = R(g)A$ for every $g \in G$, then $n = n'$ and A is invertible (in particular $R \sim R'$).

The next lemma is a general principle, which we have already used in the proof of Maschke's Theorem, and which allows us to transform K-linear maps into KG-linear maps.

Lemma 5.4

Assume $\mathrm{char}(K) \nmid |G|$. Let V, W be two KG-modules and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$, $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be the associated K-representations. If $\psi : V \longrightarrow W$ is K-linear, then the map

$$\widetilde{\psi} := \frac{1}{|G|} \sum_{g \in G} \rho_W(g) \circ \psi \circ \rho_V(g^{-1})$$

from V to W is KG-linear.

Proof: Same argument as in (3) of the proof of Maschke's Theorem: replace π by ψ and apply the fact that a G-homomorphism between representations corresponds to a KG-hmomorphism between the corresponding KG-modules. ■

Proposition 5.5

Assume $\mathrm{char}(K) \nmid |G|$. Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be two irreducible K-representations.

(a) If $\rho_V \not\sim \rho_W$ and $\psi : V \longrightarrow W$ is a K-linear map, then

$$\widetilde{\psi} = \frac{1}{|G|} \sum_{g \in G} \rho_W(g) \circ \psi \circ \rho_V(g^{-1}) = 0\,.$$

(b) Assume moreover that $K = \overline{K}$ and $\mathrm{char}(K) \nmid n := \dim_K V$. If $\psi : V \longrightarrow V$ is a K-linear map, then

$$\widetilde{\psi} := \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ \psi \circ \rho_V(g^{-1}) = \frac{\mathrm{Tr}(\psi)}{n} \cdot \mathrm{Id}_V\,.$$

Proof: Since ρ_V and ρ_W are irreducible, the associated KG-modules are simple. Moreover, by Lemma 5.4, both in (a) and (b) the map $\widetilde{\psi}$ is KG-linear. Therefore Schur's Lemma yields:

(a) $\widetilde{\psi} = 0$ since $V \not\cong W$.

(b) $\widetilde{\psi} = \lambda \cdot \mathrm{Id}_V$ for some scalar $\lambda \in K$. Therefore, on the one hand

$$\mathrm{Tr}(\widetilde{\psi}) = \frac{1}{|G|} \sum_{g \in G} \underbrace{\mathrm{Tr}\left(\rho_V(g) \circ \psi \circ \rho_V(g^{-1})\right)}_{=\mathrm{Tr}(\psi)} = \frac{1}{|G|}|G|\,\mathrm{Tr}(\psi) = \mathrm{Tr}(\psi)$$

and on the other hand

$$\mathrm{Tr}(\widetilde{\psi}) = \mathrm{Tr}(\lambda \cdot \mathrm{Id}_V) = \lambda\,\mathrm{Tr}(\mathrm{Id}_V) = n \cdot \lambda\,,$$

hence $\lambda = \frac{\mathrm{Tr}(\psi)}{n}$. ■

Next, we see that Schur's Lemma implies certain "orthogonality relations" for the entries of matrix representations.

Theorem 5.6 (Schur's Relations)

Assume $\mathrm{char}(K) \nmid |G|$. Let $Q : G \longrightarrow \mathrm{GL}_n(K)$ and $P : G \longrightarrow \mathrm{GL}_m(K)$ be irreducible matrix representations.

(a) If $P \not\sim Q$, then $\frac{1}{|G|}\sum_{g\in G} P(g)_{ri}Q(g^{-1})_{js} = 0$ for all $1 \leqslant r, i \leqslant m$ and all $1 \leqslant j, s \leqslant n$.

(b) If $\mathrm{char}(K) \nmid n$, then $\frac{1}{|G|}\sum_{g\in G} Q(g)_{ri}Q(g^{-1})_{js} = \frac{1}{n}\delta_{ij}\delta_{rs}$ for all $1 \leqslant r, i, j, s \leqslant n$.

Proof: Set $V := K^n$, $W := K^m$ and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be the K-representations induced by Q and P, respectively, as defined in Remark 1.2. Furthermore, consider the K-linear map $\psi : V \longrightarrow W$ whose matrix with respect to the standard bases of $V = K^n$ and $W = K^m$ is the elementary matrix

$$i\begin{bmatrix} & \vdots & \\ \cdots\cdots & 1 & \cdots\cdots \\ & \vdots & \end{bmatrix} =: E_{ij} \in M_{m\times n}(K)$$
$$\qquad\qquad j$$

(i.e. the unique nonzero entry of E_{ij} is its (i,j)-entry).

(a) By Proposition 5.5(a),

$$\widetilde{\psi} = \frac{1}{|G|}\sum_{g\in G}\rho_W(g)\circ\psi\circ\rho_V(g^{-1}) = 0$$

because $P \not\sim Q$, and hence $\rho_V \not\sim \rho_W$. In particular the (r,s)-entry of the matrix of $\widetilde{\psi}$ with respect to the standard bases of $V = K^n$ and $W = K^m$ is zero. Thus,

$$0 = \frac{1}{|G|}\sum_{g\in G}\left[P(g)E_{ij}Q(g^{-1})\right]_{rs} = \frac{1}{|G|}\sum_{g\in G}P(g)_{ri}\cdot 1\cdot Q(g^{-1})_{js}$$

because the unique nonzero entry of the matrix E_{ij} is its (i,j)-entry.

(b) Now we assume that $P = Q$, and hence $n = m$, $V = W$, $\rho_V = \rho_W$. Then by Proposition 5.5(b),

$$\widetilde{\psi} := \frac{1}{|G|}\sum_{g\in G}\rho_V(g)\circ\psi\circ\rho_V(g^{-1}) = \frac{\mathrm{Tr}(\psi)}{n}\cdot\mathrm{Id}_V = \begin{cases}\frac{1}{n}\cdot\mathrm{Id}_V & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases}$$

Therefore the (r,s)-entry of the matrix of $\widetilde{\psi}$ with respect to the standard basis of $V = K^n$ is

$$\frac{1}{|G|}\sum_{g\in G}\left[Q(g)E_{ij}Q(g^{-1})\right]_{rs} = \begin{cases}\left(\frac{1}{n}\cdot\mathrm{Id}_V\right)_{rs} & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases}$$

Again, because the unique nonzero entry of the matrix E_{ij} is its (i,j)-entry, it follows that

$$\frac{1}{|G|}\sum_{g\in G}Q(g)_{ri}Q(g^{-1})_{js} = \frac{1}{n}\delta_{ij}\delta_{rs}\,.$$

■

6 ℂ-Representations of Finite Abelian Groups

In this section we give an immediate application of Schur's Lemma encoding the representation theory of finite abelian groups over the field ℂ of complex numbers.

Proposition 6.1

If G is a finite abelian group, then any simple $\mathbb{C}G$-module has dimension 1.
(Equivalently, any irreducible $\mathbb{C}$-representation of G has degree 1.)

Proof: Let V be a simple $\mathbb{C}G$-module, and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be the associated $\mathbb{C}$-representation (i.e. as given by Proposition 4.3).

Claim: any $\mathbb{C}$-subspace of V is in fact a $\mathbb{C}G$-submodule.

Proof: Fix $g \in G$ and consider $\rho_V(g)$. By definition $\rho_V(g) \in \mathrm{GL}(V)$, hence it is a $\mathbb{C}$-linear endomorphism of V. We claim that it is in fact $\mathbb{C}G$-linear. Indeed, as G is abelian, $\forall\, h \in G$, $\forall\, v \in V$ we have

$$\begin{aligned}\rho_V(g)(h \cdot v) = \rho_V(g)\big(\rho_V(h)(v)\big) &= \big[\rho_V(g)\rho_V(h)\big](v) \\ &= \big[\rho_V(gh)\big](v) \\ &= \big[\rho_V(hg)\big](v) \\ &= \big[\rho_V(h)\rho_V(g)\big](v) \\ &= \rho_V(h)\big(\rho_V(g)(v)\big) \\ &= h \cdot \big(\rho_V(g)(v)\big)\end{aligned}$$

and it follows from Remark 4.4 that $\rho_V(g)$ is $\mathbb{C}G$-linear. Now, because $\mathbb{C}$ is algebraically closed, by part (b) of Schur's Lemma, there exists $\lambda_g \in \mathbb{C}$ (depending on g) such that

$$\rho_V(g) = \lambda_g \cdot \mathrm{Id}_V \ .$$

As this holds for every $g \in G$, it follows that any $\mathbb{C}$-subspace of V is G-invariant, which in terms of $\mathbb{C}G$-modules means that any $\mathbb{C}$-subspace of V is a $\mathbb{C}G$-submodule of V.

To conclude, as V is simple, we deduce from the Claim that the $\mathbb{C}$-dimension of V must be equal to 1. ■

Theorem 6.2 (*Diagonalisation Theorem*)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of an arbitrary finite group G. Fix $g \in G$. Then, there exists an ordered $\mathbb{C}$-basis B of V with respect to which

$$\big(\rho(g)\big)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{bmatrix},$$

where $n := \dim_{\mathbb{C}}(V)$ and each ε_i $(1 \leqslant i \leqslant n)$ is an $o(g)$-th root of unity in $\mathbb{C}$.

Proof: Consider the restriction of ρ to the cyclic subgroup generated by g, that is the representation

$$\rho|_{\langle g \rangle} : \langle g \rangle \longrightarrow \mathrm{GL}(V)\,.$$

By Corollary 3.6 to Maschke's Theorem, we can decompose the representation $\rho(g)|_{\langle g \rangle}$ into a direct sum of irreducible $\mathbb{C}$-representations, say

$$\rho|_{\langle g \rangle} = \rho_{V_1} \oplus \cdots \oplus \rho_{V_n}\,,$$

where $V_1, \ldots, V_n \subseteq V$ are $\langle g \rangle$-invariant. Since $\langle g \rangle$ is abelian $\dim_{\mathbb{C}}(V_i) = 1$ for each $1 \leqslant i \leqslant n$ by Proposition 6.1. Now, if for each $1 \leqslant i \leqslant n$ we choose a $\mathbb{C}$-basis $\{x_i\}$ of V_i, then there exist $\varepsilon_i \in \mathbb{C}$

$(1 \leqslant i \leqslant n)$ such that $\rho_{V_i}(g) = \varepsilon_i$ and $B := (x_1, \dots, x_n)$ is a $\mathbb{C}$-basis of V such that

$$\big(\rho(g)\big)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{bmatrix}.$$

Finally, as $g^{o(g)} = 1_G$, it follows that for each $1 \leqslant i \leqslant n$,

$$\varepsilon_i^{o(g)} = \rho_{V_i}(g)^{o(g)} = \rho_{V_i}(g^{o(g)}) = \rho_{V_i}(1_G) = 1_{\mathbb{C}}$$

and hence ε_i is an $o(g)$-th root of unity. ■

Scholium 6.3

If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of a finite abelian group, then the $\mathbb{C}$-endomorphisms $\rho(g) : V \longrightarrow V$ with g running through G are simultaneously diagonalisable.

Proof: Same argument as in the previous proof, where we may replace "$\langle g \rangle$" with the whole of G. ■