Chapter 2. The Group Algebra and Its Modules

We now introduce the concept of a KG-module, and show that this more modern approach is equivalent to the concept of a K-representation of a given finite group G. Some of the material in the remainder of these notes will be presented in terms of KG-modules. As we will soon see with our second fundamental result – Schur's Lemma – there are several advantages to this approach to representation theory.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- K denote a field of arbitrary characteristic; and
- V denote a K-vector space such that $\dim_K(V) < \infty$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all K-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

4 Modules over the Group Algebra

Lemma-Definition 4.1 (*Group algebra*)

The **group ring** KG is the ring whose elements are the K-linear combinations $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in K$, and addition and multiplication are given by

$$\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g \quad \text{and} \quad \Big(\sum_{g\in G} \lambda_g g\Big) \cdot \Big(\sum_{h\in G} \mu_h h\Big) = \sum_{g,h\in G} (\lambda_g \mu_h) gh$$

respectively. In fact KG is a K-vector space with basis G, hence a K-algebra. Thus we usually call KG the **group algebra of G over K** rather than simply *group ring*.

Note: In Definition 4.1, the field K can be replaced with a commutative ring R. E.g. if $R = \mathbb{Z}$, then $\mathbb{Z}G$ is called the *integral group ring* of G.

Proof: By definition KG is a K-vector space with basis G, and the multiplication in G is extended by K-bilinearity to the given multiplication $\cdot : KG \times KG \longrightarrow KG$. It is then straightforward to check that KG bears both the structures of a ring and of a K-vector space. Finally, axiom (A3) of K-algebras (see Appendix B) follows directly from the definition of the multiplication and the commutativity of K. ■

Remark 4.2

Clearly $1_{KG} = 1_G$, $\dim_K(KG) = |G|$, and KG is commutative if and only if G is an abelian group.

Proposition 4.3

(a) Any K-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ of G gives rise to a KG-module structure on V, where the external composition law is defined by the map

$$\begin{array}{rccl} \cdot : & KG \times V & \longrightarrow & V \\ & (\sum_{g\in G} \lambda_g g, v) & \mapsto & (\sum_{g\in G} \lambda_g g) \cdot v := \sum_{g\in G} \lambda_g \rho(g)(v) \ . \end{array}$$

(b) Conversely, every KG-module $(V, +, \cdot)$ defines a K-representation

$$\begin{array}{rccl} \rho_V : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_V(g) : V \longrightarrow V, v \mapsto \rho_V(g)(v) := g \cdot v \end{array}$$

of the group G.

Proof : (a) Since V is a K-vectore space it is equipped with an internal addition $+$ such that $(V, +)$ is an abelian group. It is then straightforward to check that the given external composition law defined above verifies the KG-module axioms.

(b) A KG-module is in particular a K-vector space for the scalar multiplication defined for all $\lambda \in K$ and all $v \in V$ by

$$\lambda v := (\underbrace{\lambda 1_G}_{\in KG}) \cdot v \, .$$

Moreover, it follows from the KG-module axioms that $\rho_V(g) \in \mathrm{GL}(V)$ and also that

$$\rho_V(g_1 g_2) = \rho_V(g_1) \circ \rho_V(g_2)$$

for all $g_1, g_2 \in G$, hence ρ_V is a group homomorphism.

See [Exercise 7, Sheet 2] for the details (Hint: use the remark below!). ■

Remark 4.4

In fact in Proposition 4.3(a) checking the KG-module axioms is equivalent to checking that for all $g, h \in G$, $\lambda \in K$ and $u, v \in V$:

(1) $(gh) \cdot v = g \cdot (h \cdot v)$;

(2) $1_G \cdot v = v$;

(4) $g \cdot (u + v) = g \cdot u + g \cdot v$;

(3) $g \cdot (\lambda v) = \lambda(g \cdot v) = (\lambda g) \cdot v$,

or in other words, that the binary operation

$$\begin{array}{rccl} \cdot : & G \times V & \longrightarrow & V \\ & (g, v) & \mapsto & g \cdot v := \rho(g)(v) \end{array}$$

is a *K-linear action of the group G on V*. Indeed, the external multiplication of KG on V is just the extension by K-linearity of the latter map. For this reason, sometimes, KG-modules are also called *G-vector spaces*. See [Exercise 6, Sheet 2] for the details.

Lemma 4.5

Two representations $\rho_1 : G \longrightarrow GL(V_1)$ and $\rho_2 : G \longrightarrow GL(V_2)$ are equivalent if and only if $V_1 \cong V_2$ as KG-modules.

Proof: If $\rho_1 \sim \rho_2$ and $\alpha : V_1 \longrightarrow V_2$ is a K-isomorphism such that $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$, then by Proposition 4.3(a) for every $v \in V_1$ and every $g \in G$ we have

$$g \cdot \alpha(v) = \rho_2(g)(\alpha(v)) = \alpha(\rho_1(g)(v)) = \alpha(g \cdot v)\,.$$

Hence α is a KG-isomorphism.
Conversely, if $\alpha : V_1 \longrightarrow V_2$ is a KG-isomorphism, then certainly it is a K-homomorphism and for each $g \in G$ and by Proposition 4.3(b) for each $v \in V_2$ we have

$$\alpha \circ \rho_1(g) \circ \alpha^{-1}(v) = \alpha(\rho_1(g)(\alpha^{-1}(v))) = \alpha(g \cdot \alpha^{-1}(v)) = g \cdot \alpha(\alpha^{-1}(v)) = g \cdot v = \rho_2(g)(v)\,,$$

hence $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$. ■

Remark 4.6 (*Dictionary*)

More generally, through Proposition 4.3, we may transport terminology and properties from KG-modules to representations and conversely.

This lets us build the following **dictionary**:

Representations		Modules
K-representation of G	$\longleftrightarrow$	KG-module
degree	$\longleftrightarrow$	K-dimension
homomorphism of representations	$\longleftrightarrow$	homomorphism of KG-modules
subrepresentation / G-invariant subspace	$\longleftrightarrow$	KG-submodule
direct sum of representations $\rho_{V_1} \oplus \rho_{V_2}$	$\longleftrightarrow$	direct sum of KG-modules $V_1 \oplus V_2$
irreducible representation	$\longleftrightarrow$	simple (= irreducible) KG-module
the trivial representation	$\longleftrightarrow$	the trivial KG-module K
the regular representation of G	$\longleftrightarrow$	the regular KG-module KG
Corollary 3.6 to Maschke's Theorem:	$\longleftrightarrow$	Corollary 3.6 to Maschke's Theorem:
If $\mathrm{char}(K) \nmid \lvert G\rvert$, then every K-representation of G is completely reducible.		If $\mathrm{char}(K) \nmid \lvert G\rvert$, then every KG-module is semisimple.
...		...

Virtually, any result, we have seen in Chapter 1, can be reinterpreted using this translation table. E.g. Property 2.4(c) tells us that the image and the kernel of homomorphisms of KG-modules are KG-submodules, ...

In this lecture, we introduce the equivalence between representations and modules for the sake of completeness. In the sequel we keep on stating results in terms of representations as much as possible. However, we will use modules when we find them more fruitful. In contrast, the M.Sc. Lecture *Representation Theory* will consistently use the module approach to representation theory.