(b) A map $f : A \to B$ between two R-algebras is called an **algebra homomorphism** iff:

 (i) f is a homomorphism of R-modules; and

 (ii) f is a ring homomorphism.

Example 15

(a) A commutative ring R itself is an R-algebra.
[The internal composition law "$\cdot$" and the external composition law "$*$" coincide in this case.]

(b) For each $n \in \mathbb{Z}_{\geqslant 1}$ the set $M_n(R)$ of $n \times n$-matrices with coefficients in a commutative ring R is an R-algebra for its usual R-module and ring structures.
[Note: in particular R-algebras need not be commutative rings in general!]

(c) Let K be a field. Then for each $n \in \mathbb{Z}_{\geqslant 1}$ the polynom ring $K[X_1, \ldots, X_n]$ is a K-algebra for its usual K-vector space and ring structure.

(d) If K is a field and V a finite-dimensional K-vector space, then $\mathrm{End}_K(V)$ is a K-algebra.

(e) $\mathbb{R}$ and $\mathbb{C}$ are $\mathbb{Q}$-algebras, $\mathbb{C}$ is an $\mathbb{R}$-algebra, ...

(f) Rings are $\mathbb{Z}$-algebras.

Definition B.2 (*Centre*)

The **centre** of an R-algebra $(A, +, \cdot, *)$ is $Z(A) := \{a \in A \mid a \cdot b = b \cdot a \ \ \forall\, b \in A\}$.

C Tensor Products of Vector Spaces

Throughout this section, we assume that K is a field.

Definition C.1 (*Tensor product of vector spaces*)

Let V, W be two finite-dimensional K-vector spaces with bases $B_V = \{v_1, \ldots, v_n\}$ and $B_W = \{w_1, \ldots, w_m\}$ ($m, n \in \mathbb{Z}_{\geqslant 0}$) respectively. The **tensor product of** V **and** W **(balanced) over** K is by definition the $(n \cdot m)$-dimensional K-vector space

$$V \otimes_K W$$

with basis $B_{V \otimes_K W} = \{v_i \otimes w_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$.

In this definition, you should understand the symbole "$v_i \otimes w_j$" as an element that depends on both v_i and w_j. The symbole "$\otimes$" itself does not have any hidden meaning, it is simply a piece of notation: we may as well write something like $x(v_i, w_j)$ instead of "$v_i \otimes w_j$", but we have chosen to write "$v_i \otimes w_j$".

Properties C.2

(a) An arbitrary element of $V \otimes_K W$ has the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{ij}(v_i \otimes w_j) \quad \text{with } \{\lambda_{ij}\}_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} \subseteq K\,.$$

(b) The binary operation

$$\begin{array}{ccc} B_V \times B_W & \longrightarrow & B_{V\otimes_K W} \\ (v_i, w_j) & \mapsto & v_i \otimes w_j \end{array}$$

can be extended by $\mathbb{C}$-linearity to

$$\begin{array}{cccc} -\otimes-: & V\times W & \longrightarrow & V\otimes_K W \\ & \big(v=\sum_{i=1}^n \lambda_i v_i, w=\sum_{i=1}^n \mu_j w_j\big) & \mapsto & v\otimes w = \sum_{i=1}^n\sum_{j=1}^m \lambda_i\mu_j(v_i\otimes w_j)\,. \end{array}$$

It follows that $\forall\ v \in V, w \in W, \lambda \in K$,

$$v\otimes(\lambda w) = (\lambda v)\otimes w = \lambda(v\otimes w)\,,$$

and $\forall\ x_1,\dots,x_r \in V, y_1,\dots y_s \in W$,

$$\Big(\sum_{i=1}^{r} x_i\Big)\otimes\Big(\sum_{j=1}^{s} y_j\Big) = \sum_{i=1}^{r}\sum_{j=1}^{s} x_i\otimes y_j\,.$$

Thus any element of $V \otimes_K W$ may also be written as a K-linear combination of elements of the form $v\otimes w$ with $v \in V, w \in W$. In other words $\{v\otimes w \mid v\in V, w\in W\}$ generates $V \otimes_K W$ (although it is not a K-basis).

(c) Up to isomorphism $V \otimes_K W$ is independent of the choice of the K-bases of V and W.

Definition C.3 (*Kronecker product*)

If $A = \big(A_{ij}\big)_{ij} \in M_n(K)$ and $B = \big(B_{rs}\big)_{rs} \in M_m(K)$ are two square matrices, then their **Kronecker product** (or **tensor product**) is the matrix

$$A\otimes B = \begin{bmatrix} A_{11}B & \cdots\cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots\cdots & A_{nn}B \end{bmatrix} \in M_{n\cdot m}(K)$$

Notice that it is clear from the above definition that $\mathrm{Tr}(A\otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$.

Example 16

E.g. the tensor product of two 2×2-matrices is of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix} \in M_4(K)\,.$$

Lemma-Definition C.4 (*Tensor product of K-endomorphisms*)

If $f_1 : V \longrightarrow V$ and $f_2 : W \longrightarrow W$ are two endomorphisms of finite-dimensional K-vector spaces V and W, then the **tensor product** of f_1 and f_2 is the K-endomorphism $f_1 \otimes f_2$ of $V \otimes_K W$ defined by

$$\begin{array}{rccl} f_1 \otimes f_2 : & V \otimes_K W & \longrightarrow & V \otimes_K W \\ & v \otimes w & \mapsto & (f_1 \otimes f_2)(v \otimes w) := f_1(v) \otimes f_2(w)\,. \end{array}$$

Furthermore, $\mathrm{Tr}(f_1 \otimes f_2) = \mathrm{Tr}(f_1)\,\mathrm{Tr}(f_2)$.

Proof: It is straightforward to check that $f_1 \otimes f_2$ is K-linear. Moreover, choosing ordered bases $B_V = \{v_1, \dots, v_n\}$ and $B_W = \{w_1, \dots, w_m\}$ of V and W respectively, it is straightforward from the definitions to check that the matrix of $f_1 \otimes f_2$ w.r.t. the ordered basis $B_{V \otimes_K W} = \{v_i \otimes w_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ is the Kronecker product of the matrices of f_1 w.r.t. B_V and of f_2 w.r.t. to B_W. The trace formula follows. ■