Throughout these exercises $(G, \cdot)$ denotes a group.

Exercise 1

Assume $G = \langle g \rangle$ is an infinite cyclic group. Prove that $0 \longrightarrow \mathbb{Z}G \xrightarrow{m_{g-1}} \mathbb{Z}G$ is a free resolution of the trivial $\mathbb{Z}G$-module, and

$$H^n(G,A) = \begin{cases} A^G & \text{if } n = 0, \\ A/\operatorname{Im}(m_{g-1}) & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 2. \end{cases}$$

Exercise 2

Let A be a $\mathbb{Z}G$-module. Prove that $\operatorname{Der}(G,A) \cong \operatorname{Hom}_{\mathbb{Z}G}(IG,A)$ via the map sending a derivation d to the homomorphism $\tilde{d}$ such that $\tilde{d}(g-1) = d(g)$, $\forall g \in G\backslash\{1\}$.

Exercise 3

Let A be a left $\mathbb{Z}G$-module, and let $A \rtimes G$ be the semi-direct product of $(A,+)$ by $(G,\cdot)$, that is, the group of all pairs $(a,g) \in A \times G$, with group law

$$(a,g)\cdot(b,h) := (a + g\cdot b, gh).$$

Let $\pi : A \rtimes G \longrightarrow G : (a,g) \mapsto g$ and let $\operatorname{Hom}'(G, A \rtimes G)$ be the set of all group homomorphisms $f : G \longrightarrow A \rtimes G$ such that $\pi \circ f = \operatorname{Id}_G$. Prove that $\operatorname{Der}(G,A)$ is in bijection with $\operatorname{Hom}'(G, A \rtimes G)$.