Throughout these exercises $(G,\cdot)$ denotes a group and K a commutative ring (of course associative and unital).

Exercise 1

(a) Let M and N be KG-modules. Prove that:

(i) M^G is the largest submodule of M on which G acts trivially;

(ii) M_G is the largest quotient of M on which G acts trivially;

(iii) $\mathrm{Hom}_K(M,N)^G = \mathrm{Hom}_{KG}(M,N)$;

(iv) $(M\otimes_K N)_G \cong M\otimes_{KG} N$.

(b) Prove that if G is finite, then $(KG)^G = \langle\sum_{g\in G} g\rangle_K$ and if G is infinite, then $(KG)^G = 0$.

Exercise 2

(a) Prove that there is an isomorphism of abelian groups $(IG/(IG)^2,+)\cong(G_{ab},\cdot)$, where G_{ab} denotes the abelianisation of G;

(b) Assume A is a trivial $\mathbb{Z}G$-module. Prove that there are group isomorphisms

(i) $H_1(G,A)\cong IG\otimes_{\mathbb{Z}G} A\cong IG/(IG)^2\otimes_{\mathbb{Z}G} A\cong IG/(IG)^2\otimes_{\mathbb{Z}} A\cong G_{ab}\otimes_{\mathbb{Z}} A$;

(ii) $H^1(G,A)\cong \mathrm{Hom}_{\mathbb{Z}G}(IG,A)\cong\mathrm{Hom}_{\mathbb{Z}G}(IG/(IG)^2,A)$
$\cong\mathrm{Hom}_{\mathbb{Z}}(IG/(IG)^2,A)\cong\mathrm{Hom}_{\mathbb{Z}}(G_{ab},A)\cong\mathrm{Hom}_{\mathbf{Grp}}(G,A)$.

Exercise 3

Let A be a KG-module.

(a) Prove that if F is a free $\mathbb{Z}G$-module, then

$$\mathrm{Hom}_{KG}(K\otimes_{\mathbb{Z}} F,A)\cong\mathrm{Hom}_{\mathbb{Z}G}(F,A)\,.$$

[Hint: Given a $\mathbb{Z}$-basis X of F, prove that $K\otimes_{\mathbb{Z}} F$ is also free and describe a K-basis.]

(b) Prove that $\mathrm{Ext}^n_{KG}(K,A)\cong\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},A)$ for every $n\geqslant 0$.
[Hint: Given a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, construct a free resolution of K as a KG-module, using the "same" bases.]