Cohomology of Groups — Exercise Sheet 6 TU Kaiserslautern

Jun.-Prof. Dr. Caroline Lassueur **Bernhard Böhmler**

Due date: **Monday, 31st of May 2021, 18:00** SS 2021

Throughout these exercises R denotes an associative and unital ring.

Exercise 1

(a) Prove that if $\mathrm{Ext}^1_R(M,N) = 0$, then any s.e.s. $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$ of R-modules splits.

(b) Let P be an R-module. Prove that the following assertions are equivalent:

  (i) P is projective;

  (ii) $\mathrm{Ext}^n_R(P,N) = 0$ for every $n \geqslant 1$ and each R-module N; and

  (iii) $\mathrm{Ext}^1_R(P,N) = 0$ for each R-module N.

Exercise 2

Let A be a $\mathbb{Z}$-module and let p be a positive prime number. Prove that:

(a) $\mathrm{Tor}^{\mathbb{Z}}_\bullet(A, \mathbb{Z}/p\mathbb{Z})$ is the homology of the complex $0 \longrightarrow A \xrightarrow{\cdot p} A \longrightarrow 0$;

(b) $\mathrm{Tor}^{\mathbb{Z}}_0(A, \mathbb{Z}/p\mathbb{Z}) \cong A/pA$,

$\mathrm{Tor}^{\mathbb{Z}}_1(A, \mathbb{Z}/p\mathbb{Z}) \cong A_p := \{a \in A \mid p \cdot a = 0\}$,

$\mathrm{Tor}^{\mathbb{Z}}_n(A, \mathbb{Z}/p\mathbb{Z}) = 0$ if $n \geqslant 2$;

(c) $\mathrm{Ext}^\bullet_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, A)$ is the cohomology of the complex $0 \longrightarrow A \xrightarrow{\cdot p} A \longrightarrow 0$;

(d) $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, A) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, A) \cong A_p$,

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, A) \cong A/pA$,

$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, A) = 0$ if $n \geqslant 2$.

Exercise 3

Consider the following commutative diagram of R-modules with exact rows:

$$\begin{array}{ccccccc}
A' & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & A'' & \longrightarrow & 0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\
B' & \xrightarrow[\varphi]{} & B & \xrightarrow[\psi]{} & B'' & \longrightarrow & 0
\end{array}$$

Prove that there exists a morphism $h \in \mathrm{Hom}_R(A'', B'')$ such that $h \circ \beta = \psi \circ g$. Moreover, if f and g are isomorphisms, then so is h.