Jun.-Prof. Dr. Caroline Lassueur **Bernhard Böhmler**
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Throughout these exercises R denotes an associative and unital ring.

Definition. A chain complex $\mathbf{C}_\bullet$ of R-modules is called **split exact** if it is exact and if moreover for each $n \in \mathbb{Z}$, $Z_n := Z_n(\mathbf{C}_\bullet)$ is a direct summand of C_n, i.e. $C_n = Z_n \oplus U_n$ for some R-module U_n.

Exercise 1

Let $(\mathbf{C}_\bullet, \mathbf{d}_\bullet)$ be a chain complex of R-modules.

(a) With the notation of the definition, prove that:

(i) If $\mathbf{C}_\bullet$ is split exact, d_n induces an isomorphism $U_n \xrightarrow{\cong} Z_{n-1}$ for all $n \in \mathbb{Z}$.

(ii) The inverse of the isomorphism of (a) induces an R-homomorphism $s_n : C_{n-1} \longrightarrow C_n$ such that $\ker(s_n) = U_{n-1}$ and $\operatorname{Im}(s_n) = U_n$.

(iii) $\mathbf{C}_\bullet$ is split exact if and only if $\mathrm{Id}_{\mathbf{C}_\bullet}$ is homotopic to the zero chain map.

(b) Prove that $(\mathbf{C}_\bullet, \mathbf{d}_\bullet)$ is split exact if and only if $\mathbf{C}_\bullet$ is exact and there are R-homomorphisms $s_n : C_n \longrightarrow C_{n+1}$ such that $d_{n+1}s_nd_{n+1} = d_{n+1}$.

(Hint: For the sufficient condition, prove $\ker(sd) \subseteq \operatorname{Im}(ds)$ (where we omit the indices for clarity).)

(c) For $R \in \{\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}\}$ prove that the following complex of R-modules is acyclic but not split exact:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \cdots$$

Exercise 2

Consider the two non-negative chain complexes of $\mathbb{Z}$-modules

$$\mathbf{P}_\bullet := (0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z}) \quad \text{and} \quad \mathbf{Q}_\bullet := (0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z})$$

where the rightmost module is assumed to be in degree zero. Let

$$f : H_0(\mathbf{P}_\bullet) = \mathbb{Z}/4\mathbb{Z} \longrightarrow H_0(\mathbf{Q}_\bullet) = \mathbb{Z}/2\mathbb{Z}$$

be the unique non-zero $\mathbb{Z}$-linear map.

(a) Find all possible chain maps $\varphi_\bullet : \mathbf{P}_\bullet \longrightarrow \mathbf{Q}_\bullet$ lifting f.

(b) Construct homotopies between the different liftings of part (a).

Exercise 3

Prove the Horseschoe Lemma. [Hint: Proceed by induction on n and use the Snake Lemma.]