COHOMOLOGY OF GROUPS — EXERCISE SHEET 3 **TU KAISERSLAUTERN**

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This Exercise Sheet is a recap sheet on module theory, with focus on exact sequences, the Hom functors and tensor products. Throughout, R and S denote unital associative rings.

EXERCISE 1
Prove that if $\varphi : M \longrightarrow N$ is a homomorphism of R-modules, then there is always an exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow M \xrightarrow{\varphi} N \longrightarrow \mathrm{Coker}(\varphi) \longrightarrow 0\,.$$

Compute the sequences associated with the following $\mathbb{Z}$-homomorphisms:

(i) $\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$ (i.e. external multiplication by a prime number p in $\mathbb{Z}$);

(ii) $\mathbb{Z}/15\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z}/15\mathbb{Z}$ (i.e. external multiplication by $3 \in \mathbb{Z}$).

EXERCISE 2
Let Q be an R-module and let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of R-modules.

(a) Find a counterexample in which the functor $\mathrm{Hom}_R(Q, -)$ does not preserve surjectivity, i.e find a surjective R-homomorphism $g : B \longrightarrow C$ such that the induced homomorphism $g_* : \mathrm{Hom}_R(Q, B) \longrightarrow \mathrm{Hom}_R(Q, C)$ is not surjective.

(b) Prove that the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(C, Q) \xrightarrow{g^*} \mathrm{Hom}_R(B, Q) \xrightarrow{f^*} \mathrm{Hom}_R(A, Q)$$

is exact.

Find a counterexample of an injective R-homomorphism $f : A \longrightarrow B$ such that the induced homomorphism $f^* : \mathrm{Hom}_R(B, Q) \longrightarrow \mathrm{Hom}_R(A, Q)$ is not surjective.

EXERCISE 3 (**Extension of scalars**)

(a) **Extension of scalars**: Prove that for every left R-module M, the tensor product $S \otimes_R M$ can be endowed with a left S-module structure via $x \cdot (s \otimes m) = xs \otimes m$, $\forall\, x, s \in S, m \in M$.

(b) Prove that the map $\iota : M \longrightarrow S \otimes_R M : m \mapsto \iota(m) = 1 \otimes m$ is an R-homomorphism, for every left R-module M.

(c) **Universal property of the extension of scalars**: Prove that for every left S-module N and for every R-homomorphism $g : M \longrightarrow N$ (where N is seen as an R-module via restriction of scalars), there exists a unique S-homomorphism $\widetilde{g} : S \otimes_R M \longrightarrow N$ such that $\widetilde{g} \circ \iota = g$, that is such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\;g\;} & N \\ {\scriptstyle\iota}\big\downarrow & \circlearrowleft & \nearrow {\scriptstyle\widetilde{g}} \\ S \otimes_R M & & \end{array}$$

(d) Prove that if M is a free left R-module with basis $\{e_i\}_{i\in I}$, then $S \otimes_R M$ is a free S-module with basis $\{1 \otimes e_i\}_{i\in I}$.

(e) Assume $R \subseteq S$ is an extension of commutative rings such that S is a free R-module of finite rank n. Prove that $S \otimes_R M$ is R-isomorphic to a direct sum of n copies of M.

(f) If $S \cong R/I$ is a quotient of R by a two-sided ideal I and $f : R \longrightarrow R/I$ is the quotient morphism, recall that $S \otimes_R M = R/I \otimes_R M \cong M/IM$ and deduce that the map ι is not necessarily injective.