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Due date: **Monday, 3rd of May 2021, 18:00** SS 2021

Exercise 1

(a) Let X and Y be two sets with the same cardinality. Prove that $F_X \cong F_Y$.
[Notice that the converse holds as well. You can try to find a proof too, but arguments are more involved.]

(b) Prove that in a free group, every equivalence class of words contains a unique reduced word.

(c) How many reduced words of length $\ell \geqslant 1$ are there in a free group of rank $r \in \mathbb{Z}_{>0}$?

Exercise 2

Let $G = \langle x, y \,|\, x^2 = y^2 = (xy)^2 \rangle$. Prove that G is a finite group, determine its order and identify this group up to isomorphism.
[Hint. Draw the Cayley graph of G. Consider $Z(G)$ and $G/Z(G)$.]

Exercise 3

Prove Theorem 2.7 through the following steps. Set $r := st$ and let m be the order of r. Set $H := \langle r \rangle \cong C_m$ and $C := \langle s \rangle \cong C_2$.

1. Prove that $m \geqslant 2$ and $srs^{-1} = r^{-1}$.
2. Prove that $G = H \rtimes C = D_{2m}$.
3. By 2. and Example 4, G admits the presentation $\langle r, s \mid r^m = 1, s^2 = 1, srs^{-1} = 1 \rangle$. Apply the universal property of presentations twice to prove that G also admits the presentation $\langle s, t \mid s^2 = 1, t^2 = 1, (st)^m = 1 \rangle$.