Exercise 1

Let K be a field.

(a) Prove that

$$\mathrm{GL}_n(K) = \mathrm{SL}_n(K) \rtimes \left\{ \mathrm{diag}(\lambda, 1, \ldots, 1) \in \mathrm{GL}_n(K) \mid \lambda \in K^\times \right\},$$

where $\mathrm{diag}(\lambda, 1, \ldots, 1)$ is the diagonal matrix with (ordered) diagonal entries $\lambda, 1, \ldots, 1$. Describe the action.

(b) Let

$$B := \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathrm{GL}_n(K) \right\} \quad (= \text{ upper triangular matrices}),$$

$$U := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}_n(K) \right\} \quad (= \text{ upper unitriangular matrices}),$$

$$T := \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \in \mathrm{GL}_n(K) \right\} \quad (= \text{ diagonal matrices}).$$

Prove that B is a semi-direct product of U by T, that is, $B = U \rtimes T$. Describe the action.

Exercise 2

(a) Prove that $D_8 \cong V_4 \rtimes C_2$, where V_4 is the Klein-four group. Describe the action of C_2 on V_4.

(b) Prove that $\mathfrak{S}_4 \cong V_4 \rtimes \mathfrak{S}_3$. Deduce that a Sylow 2-subgroup of $\mathfrak{S}_4$ is isomorphic to D_8.

(c) Construct all semi-direct products of C_3 by C_3 up to isomorphism.

(d) Identify the group described in Example 1.6(2) when $m = 7$, $n = 3$ and $k = 2$. (Hint: is it an abelian group?)