Chapter 2. Background Material: Module Theory

The aim of this chapter is to recall the basics of the theory of modules, which we will use throughout. We review elementary constructions such as quotients, direct sum, direct products, exact sequences, free/projective/injective modules and tensor products, where we emphasise the approach via universal properties. Particularly important for the forthcoming homological algebra and cohomology of groups are the notions of free and, more generally, of projective modules.

Throughout this chapter we let R and S denote rings, and unless otherwise specified, all rings are assumed to be *unital* and *associative*.

Most results are stated without proof, as they have been studied in the B.Sc. lecture *Commutative Algebra*. As further reference I recommend for example:

Reference:

[Rot10] J. J. Rotman, *Advanced modern algebra. 2nd ed.*, Providence, RI: American Mathematical Society (AMS), 2010.

3 Modules, Submodules, Morphisms

Definition 3.1 (*Left R-module, right R-module, (R,S)-bimodule, homomorphism of modules*)

(a) A **left R-module** is an abelian group $(M,+)$ endowed with a **scalar multiplication** (or **external composition law**) $\cdot : R\times M \longrightarrow M, (r,m)\mapsto r\cdot m$ such that the map

$$\begin{array}{rccl} \lambda: & R & \longrightarrow & \mathrm{End}(M) \\ & r & \mapsto & \lambda(r) := \lambda_r : M \longrightarrow M, m\mapsto r\cdot m\,, \end{array}$$

is a ring homomorphism. By convention, when no confusion is to be made, we will simply write "R-module" to mean "left R-module", and rm instead of $r\cdot m$.

(a') A **right R-module** is defined analogously using a scalar multiplication $\cdot : M\times R \longrightarrow M$, $(m,r)\mapsto m\cdot r$ on the right-hand side.

(a") If S is a second ring, then an **(R,S)-bimodule** is an abelian group $(M,+)$ which is both a left R-module and a right S-module, and which satisfies the axiom

$$r\cdot(m\cdot s) = (r\cdot m)\cdot s \qquad \forall\, r\in R, \forall\, s\in S, \forall\, m\in M\,.$$

(b) An **R-submodule** of an R-module M is a subgroup $N \leqslant M$ such that $r \cdot n \in N$ for every $r \in R$ and every $n \in N$. (Similarly for right modules and bimodules.)

(c) A **(homo)morphism** of R-modules (or an **R-linear map**, or an **R-homomorphism**) is a map of R-modules $\varphi : M \longrightarrow N$ such that:

  (i) φ is a group homomorphism; and

  (ii) $\varphi(r \cdot m) = r \cdot \varphi(m)\ \forall\ r \in R, \forall\ m \in M$.

A bijective homomorphism of R-modules is called an **isomorphism** (or an R-isomorphism), and we write $M \cong N$ if there exists an R-isomorphism between M and N.
An injective (resp. surjective) homomorphism of R-modules is sometimes called a **monomorphism** (resp. epimorphism) and we sometimes denote it with a *hook arrow* "$\hookrightarrow$" (resp. a *two-head arrow* "$\twoheadrightarrow$").
(Similarly for right modules and bimodules.)

Notation: We let ${}_R\mathbf{Mod}$ denote the category of left R-modules (with R-linear maps as morphisms), we let $\mathbf{Mod}_R$ denote the category of right R-modules (with R-linear maps as morphisms), and we let ${}_R\mathbf{Mod}_S$ denote the category of (R, S)-bimodules (with (R, S)-linear maps as morphisms). For the language of category theory, see the Appendix.

Convention: From now on, unless otherwise stated, we will always work with left modules.

Example 5

(a) Vector spaces over a field K are K-modules, and conversely.

(b) Abelian groups are $\mathbb{Z}$-modules, and conversely.

(c) If the ring R is commutative, then any right module can be made into a left module, and conversely.

(d) If $\varphi : M \longrightarrow N$ is a morphism of R-modules, then the kernel $\ker(\varphi)$ of φ is an R-submodule of M and the image $\mathrm{Im}(\varphi) := \varphi(M)$ of f is an R-submodule of N.

Notation 3.2

Given R-modules M and N, we set $\mathrm{Hom}_R(M, N) := \{\varphi : M \longrightarrow N \mid \varphi \text{ is an } R\text{-homomorphism}\}$.
This is an abelian group for the pointwise addition of functions:

$$\begin{array}{rccl} + : & \mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(M,N) & \longrightarrow & \mathrm{Hom}_R(M,N) \\ & (\varphi, \psi) & \mapsto & \varphi + \psi : M \longrightarrow N, m \mapsto \varphi(m) + \psi(m)\,. \end{array}$$

In case $N = M$, we write $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ for the set of endomorphisms of M and $\mathrm{Aut}_R(M)$ for the set of automorphisms of M, i.e. the set of invertible endomorphisms of M.

Exercise [Exercise 1, Exercise Sheet 3]

Let M, N be R-modules. Prove that:

(a) $\mathrm{End}_R(M)$, endowed with the usual composition and sum of functions, is a ring.

(b) If R is commutative then the abelian group $\mathrm{Hom}_R(M, N)$ is a left R-module.

Lemma-Definition 3.3 (*Quotients of modules*)

Let U be an R-submodule of an R-module M. The quotient group M/U can be endowed with the structure of an R-module in a natural way:

$$\begin{aligned} R \times M/U &\longrightarrow M/U \\ (r, m + U) &\longmapsto r \cdot m + U \end{aligned}$$

The canonical map $\pi : M \longrightarrow M/U, m \mapsto m + U$ is R-linear.

Proof: Direct calculation. ∎

Theorem 3.4

(a) **Universal property of the quotient**: Let $\varphi : M \longrightarrow N$ be a homomorphism of R-modules. If U is an R-submodule of M such that $U \subseteq \ker(\varphi)$, then there exists a unique R-module homomorphism $\overline{\varphi} : M/U \longrightarrow N$ such that $\overline{\varphi} \circ \pi = \varphi$, or in other words such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle \pi}\downarrow & \circlearrowleft & \nearrow {\scriptstyle \exists!\,\overline{\varphi}} \\ M/U & & \end{array}$$

Concretely, $\overline{\varphi}(m + U) = \varphi(m) \ \forall\ m + U \in M/U$.

(b) **1st isomorphism theorem**: With the notation of (a), if $U = \ker(\varphi)$, then

$$\overline{\varphi} : M/\ker(\varphi) \longrightarrow \mathrm{Im}(\varphi)$$

is an isomorphism of R-modules.

(c) **2nd isomorphism theorem**: If U_1, U_2 are R-submodules of M, then so are $U_1 \cap U_2$ and $U_1 + U_2$, and there is an an isomorphism of R-modules

$$(U_1 + U_2)/U_2 \cong U_1/U_1 \cap U_2 \,.$$

(d) **3rd isomorphism theorem**: If $U_1 \subseteq U_2$ are R-submodules of M, then there is an an isomorphism of R-modules

$$\left(M/U_1\right) / \left(U_2/U_1\right) \cong M/U_2 \,.$$

(e) **Correspondence theorem**: If U is an R-submodule of M, then there is a bijection

$$\begin{array}{ccc} \{X \ R\text{-submodule of } M \mid U \subseteq X\} & \longleftrightarrow & \{R\text{-submodules of } M/U\} \\ X & \mapsto & X/_U \\ \pi^{-1}(Z) & \leftarrow\!\shortmid & Z\,. \end{array}$$

Proof: We assume it is known from the "Einführung in die Algebra" that these results hold for abelian groups and morphisms of abelian groups. Exercise: check that they carry over to the R-module structure. ■

Definition 3.5 (*Cokernel, coimage*)

Let $\varphi \in \mathrm{Hom}_R(M, N)$. Then, the **cokernel** of φ is the quotient R-module $N/_{\mathrm{Im}\,\varphi}$, and the **coimage** of φ is the quotient R-module $M/_{\ker\varphi}$.

4 Direct products and direct sums

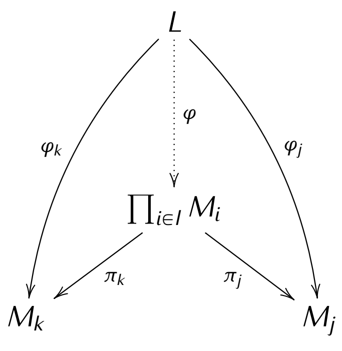
Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then the abelian group $\prod_{i\in I} M_i$, that is the product of $\{M_i\}_{i\in I}$ seen as a family of abelian groups, becomes an R-module via the following external composition law:

$$\begin{aligned} R \times \prod_{i\in I} M_i &\longrightarrow \prod_{i\in I} M_i \\ \big(r, (m_i)_{i\in I}\big) &\longmapsto \big(r \cdot m_i\big)_{i\in I}\,. \end{aligned}$$

Furthermore, for each $j \in I$, we let $\pi_j : \prod_{i\in I} M_i \longrightarrow M_j$ denotes the j-th projection from the product to the module M_j.

Proposition 4.1 (Universal property of the direct product)

If $\{\varphi_i : L \longrightarrow M_i\}_{i\in I}$ is a collection of R-linear maps, then there exists a unique morphism of R-modules $\varphi : L \longrightarrow \prod_{i\in I} M_i$ such that $\pi_j \circ \varphi = \varphi_j$ for every $j \in I$.



In other words

$$\begin{aligned} \mathrm{Hom}_R\Big(L, \prod_{i\in I} M_i\Big) &\longrightarrow \prod_{i\in I} \mathrm{Hom}_R(L, M_i) \\ f &\longmapsto \big(\pi_i \circ f\big)_i \end{aligned}$$

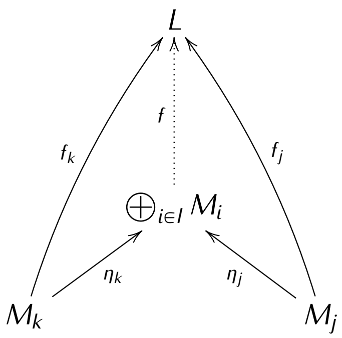
is an isomorphism of abelian groups.

Proof: Exercise 2, Exercise Sheet 3. ■

Now let $\bigoplus_{i\in I} M_i$ be the subgroup of $\prod_{i\in I} M_i$ consisting of the elements $(m_i)_{i\in I}$ such that $m_i = 0$ almost everywhere (i.e. $m_i = 0$ exept for a finite subset of indices $i \in I$). This subgroup is called the **direct sum** of the family $\{M_i\}_{i\in I}$ and is in fact an R-submodule of the product. For each $j \in I$, we let $\eta_j : M_j \longrightarrow \bigoplus_{i\in I} M_i$ denote the canonical injection of M_j in the direct sum.

Proposition 4.2 (Universal property of the direct sum)

If $\{f_i : M_i \longrightarrow L\}_{i\in I}$ is a collection of R-linear maps, then there exists a unique morphism of R-modules $\varphi : \bigoplus_{i\in I} M_i \longrightarrow L$ such that $f \circ \eta_j = f_j$ for every $j \in I$.



In other words

$$\mathrm{Hom}_R\Big(\bigoplus_{i\in I} M_i, L\Big) \longrightarrow \prod_{i\in I} \mathrm{Hom}_R(M_i, L)$$
$$f \longmapsto (f\circ\eta_i)_i$$

is an isomorphism of abelian groups.

Proof: Exercise 2, Exercise Sheet 3. ■

Remark 4.3

It is clear that if $|I| < \infty$, then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$.

The direct sum as defined above is often called an *external* direct sum. This relates as follows with the usual notion of *internal* direct sum:

Definition 4.4 (*"Internal" direct sums*)

Let M be an R-module and N_1, N_2 be two R-submodules of M. We write $M = N_1 \oplus N_2$ if every $m \in M$ can be written in a unique way as $m = n_1 + n_2$, where $n_1 \in N_1$ and $n_2 \in N_2$.

In fact $M = N_1 \oplus N_2$ (internal direct sum) if and only if $M = N_1 + N_2$ and $N_1 \cap N_2 = \{0\}$.

Proposition 4.5

If N_1, N_2 and M are as above and $M = N_1 \oplus N_2$ then the homomorphism of R-modules

$$\begin{array}{rccl} \varphi: & M & \longrightarrow & N_1 \times N_2 = N_1 \oplus N_2 \quad \text{(external direct sum)} \\ & m = n_1 + n_2 & \mapsto & (n_1, n_2)\,, \end{array}$$

is an isomorphism of R-modules.

The above generalises to arbitrary internal direct sums $M = \bigoplus_{i\in I} N_i$.

5 Exact Sequences

Definition 5.1 (*Exact sequence*)

A sequence $L \xrightarrow{\varphi} M \xrightarrow{\psi} N$ of R-modules and R-linear maps is called **exact (at** M**)** if $\operatorname{Im}\varphi = \ker\psi$.

Remark 5.2 (*Injectivity/surjectivity/short exact sequences*)

(a) $L \xrightarrow{\varphi} M$ is injective $\Longleftrightarrow$ $0 \longrightarrow L \xrightarrow{\varphi} M$ is exact at L.

(b) $M \xrightarrow{\psi} N$ is surjective $\Longleftrightarrow$ $M \xrightarrow{\psi} N \longrightarrow 0$ is exact at N.

(c) $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is exact (i.e. at L, M and N) if and only if φ is injective, ψ is surjective and ψ induces an isomorphism $\overline{\psi} : M/_{\operatorname{Im}\varphi} \longrightarrow N$.
Such a sequence is called a **short exact sequence** (s.e.s. in short).

(d) If $\varphi \in \operatorname{Hom}_R(L, M)$ is an injective morphism, then there is a s.e.s.

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\pi} \operatorname{coker}(\varphi) \longrightarrow 0$$

where π is the canonical projection.

(d) If $\psi \in \operatorname{Hom}_R(M, N)$ is a surjective morphism, then there is a s.e.s.

$$0 \longrightarrow \ker(\varphi) \xrightarrow{i} M \xrightarrow{\psi} N \longrightarrow 0\,,$$

where i is the canonical injection.

Proposition 5.3

Let Q be an R-module. Then the following holds:

(a) $\operatorname{Hom}_R(Q, -) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact covariant functor. In other words, if $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is a s.e.s of R-modules, then the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(Q, L) \xrightarrow{\varphi_*} \operatorname{Hom}_R(Q, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(Q, N)$$

is an exact sequence of abelian groups. (Here $\varphi_* := \operatorname{Hom}_R(Q, \varphi)$, that is $\varphi_*(\alpha) = \varphi \circ \alpha$ and similarly for ψ_*.)

(b) $\operatorname{Hom}_R(-, Q) : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a *left* exact contravariant functor. In other words, if

$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ is a s.e.s of R-modules, then the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(N,Q) \xrightarrow{\psi^*} \mathrm{Hom}_R(M,Q) \xrightarrow{\varphi^*} \mathrm{Hom}_R(L,Q)$$

is an exact sequence of abelian groups. (Here $\varphi^* := \mathrm{Hom}_R(\varphi, Q)$, that is $\varphi^*(\alpha) = \alpha \circ \varphi$ and similarly for ψ^*.)

Proof: One easily checks that $\mathrm{Hom}_R(Q,-)$ and $\mathrm{Hom}_R(-,Q)$ are functors.

(a) · **Exactness at** $\mathrm{Hom}_R(Q,L)$**:** Clear.
· **Exactness at** $\mathrm{Hom}_R(Q,M)$**:** We have

$$\begin{aligned} \beta \in \ker \psi_* &\iff \psi \circ \beta = 0 \\ &\iff \mathrm{Im}\,\beta \subset \ker \psi \\ &\iff \mathrm{Im}\,\beta \subset \mathrm{Im}\,\varphi \\ &\iff \forall q \in Q, \exists!\ l_q \in L \text{ such that } \beta(q) = \varphi(l_q) \\ &\iff \exists \text{ a map } \lambda : Q \longrightarrow L \text{ which sends } q \text{ to } l_q \text{ and such that } \varphi \circ \lambda = \beta \\ &\overset{\varphi \text{ inj}}{\iff} \exists\, \lambda \in \mathrm{Hom}_R(Q,L) \text{ which send } q \text{ to } l_q \text{ and such that } \varphi \circ \lambda = \beta \\ &\iff \beta \in \mathrm{Im}\,\varphi_*. \end{aligned}$$

(b) Exercise 5, Exercise Sheet 3. ■

Remark 5.4

Notice that $\mathrm{Hom}_R(Q,-)$ and $\mathrm{Hom}_R(-,Q)$ are not *right* exact in general. See Exercise 5, Exercise Sheet 3.

Lemma 5.5 (*The snake lemma*)

Suppose we are given the following commutative diagram of R-modules and R-module homomorphisms with exact rows:

$$\begin{array}{ccccccc} & & L & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & N & \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & \\ 0 & \longrightarrow & L' & \xrightarrow{\varphi'} & M' & \xrightarrow{\psi'} & N' & \end{array}$$

Then the following hold:

(a) There exists an exact sequence

$$\ker f \xrightarrow{\varphi} \ker g \xrightarrow{\psi} \ker h \xrightarrow{\delta} \mathrm{coker}\, f \xrightarrow{\overline{\varphi'}} \mathrm{coker}\, g \xrightarrow{\overline{\psi'}} \mathrm{coker}\, h,$$

where $\overline{\varphi'}, \overline{\psi'}$ are the morphisms induced by the universal property of the quotient, and $\delta(n) = \pi_L \circ {\varphi'}^{-1} \circ g \circ \psi^{-1}(n)$ for every $n \in \ker(h)$ (here $\pi_L : L \longrightarrow \mathrm{coker}(f)$ is the canonical homomorphism). The map δ is called the **connecting homomomorphism**.

(b) If $\varphi : L \longrightarrow M$ is injective, then $\varphi|_{\ker f} : \ker f \longrightarrow \ker g$ is injective.

(c) If $\psi' : M' \longrightarrow N'$ is surjective, then $\overline{\psi'} : \mathrm{coker}\, g \longrightarrow \mathrm{coker}\, h$ is surjective.

Proof: (a) First, we check that δ is well-defined. Let $n \in \ker h$ and choose two preimages $m_1, m_2 \in M$ of n under ψ. Hence $m_1 - m_2 \in \ker \psi = \mathrm{Im}\,\varphi$. Thus, there exists $l \in L$ such that $m_1 = \varphi(l) + m_2$.

Then, we have

$$g(m_1) = g \circ \varphi(l) + g(m_2) = \varphi' \circ f(l) + g(m_2).$$

Since $n \in \ker h$, for $i \in \{1, 2\}$ we have

$$\psi' \circ g(m_i) = h \circ \psi(m_i) = h(n) = 0,$$

so that $g(m_i) \in \ker \psi' = \operatorname{Im} \varphi'$. Therefore, there exists $l'_i \in L'$ such that $\varphi'(l'_i) = g(m_i)$. It follows that

$$g(m_2) = \varphi'(l'_2) = \varphi' \circ f(l) + \varphi'(l'_1).$$

Since φ' is injective, we obtain $l'_2 = f(l) + l'_1$. Hence, l'_1 and l'_2 have the same image in $\operatorname{coker} f$. Therefore, δ is well-defined.

We now want to check the exactness at $\ker h$. Let $m \in \ker g$. Then $g(m) = 0$, so that $\delta\psi(m) = 0$ and thus $\operatorname{Im} \psi|_{\ker h} \subset \ker \delta$. Conversely, let $m \in \ker \delta$. With the previous notation, this means that $\overline{l'_1} = 0$, and thus $l'_1 = f(\tilde{l})$ for some $\tilde{l} \in L$. We have

$$g \circ \varphi(\tilde{l}) = \varphi' \circ f(\tilde{l}) = \varphi'(l'_1) = g(m_1).$$

Hence, $m_1 - \varphi(\tilde{l}) \in \ker g$. It remains to check that this element is sent to n by ψ. We get

$$\psi\big(m_1 - \varphi(\tilde{l})\big) = \psi(m_1) - \psi \circ \varphi(\tilde{l}) = \psi(m_1) = n.$$

Hence $\operatorname{Im} \psi|_{\ker h} = \ker \delta$.

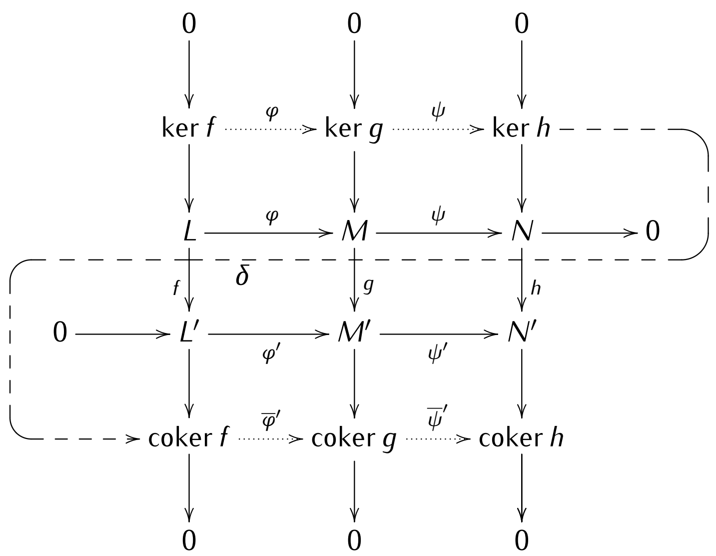
The fact that δ is an R-homomorphism, and the exactness at the other points are checked in a similar fashion.

(b) Is obvious.

(c) Is a a direct consequence of the universal property of the quotient. ■

Remark 5.6

The name of the lemma comes from the following diagram



If fact the snake lemma holds in any abelian category. In particular, it holds for the categories of chain and cochain complexes, which we will study in Chapter 3.

Lemma-Definition 5.7

A s.e.s. $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ of R-modules is called **split** iff it satisfies the following equivalent conditions:

(a) There exists an R-linear map $\sigma : N \longrightarrow M$ such that $\psi \circ \sigma = \mathrm{id}_N$ (σ is called a **section** for ψ).

(b) There exists an R-linear map $\rho : M \longrightarrow L$ such that $\rho \circ \varphi = \mathrm{id}_L$ (ρ is called a **retraction** for φ).

(c) The submodule $\mathrm{Im}\,\varphi = \ker\psi$ is a *direct summand* of M, that is there exists a submodule M' of M such that $M = \mathrm{Im}\,\varphi \oplus M'$.

Proof: Exercise. ∎

Example 6

The sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

defined by $\varphi([1]) = ([1],[0])$ and π is the canonical projection into the cokernel of φ is split but the squence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

defined by $\varphi([1]) = ([2])$ and π is the canonical projection onto the cokernel of φ is not split.

6 Free, Injective and Projective Modules

Free modules

Definition 6.1 (*Generating set / R-basis / free R-module*)

Let M be an R-module and $X \subseteq M$ be a subset.

(a) M is said to be **generated** by X if every element of M can be written as an R-linear combination $\sum_{x\in X} \lambda_x x$, that is with $\lambda_x \in R$ almost everywhere 0.

(b) X is an R-**basis** (or a **basis**) if X generates M and if every element of M can be written *in a unique way* as an R-linear combination $\sum_{x\in X} \lambda_x X$ (i.e. with $\lambda_s \in R$ almost everywhere 0).

(c) M is called **free** if it admits an R-basis.
Notation: In this case we write $M = \bigoplus_{x\in X} Rx \cong \bigoplus_{x\in X} R =: R^{(X)}$.

Remark 6.2

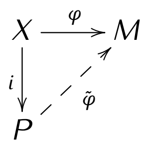
(a) When we write the sum $\sum_{x\in X} \lambda_x X$, we always assume that the λ_s are 0 almost everywhere.

(b) Let X be a generating set for M. Then, X is a basis of M if and only if S is R-linearly independent.

(c) If R is a field, then every R-module is free. (R-vector spaces.)

Proposition 6.3 (*Universal property of free modules*)

Let P be a free R-module with basis X and let $i : X \hookrightarrow P$ be the inclusion map. For every R-module M and for every *map* (of sets) $\varphi : X \longrightarrow M$, there exists a unique morphism of R-modules $\tilde{\varphi} : P \longrightarrow M$ such that the following diagram commutes



Proof: If $P \ni m = \sum_{x \in X} \lambda_x x$ (unique expression), then we set $\tilde{\varphi}(m) = \sum_{x \in X} \lambda_x \varphi(x)$. It is then easy to check $\tilde{\varphi}$ has the required properties. ■

Proposition 6.4 (*Properties of free modules*)

(a) Every R-module M is isomorphic to a quotient of a free R-module.

(b) If P is a free R-module, then $\mathrm{Hom}_R(P, -)$ is an exact functor.

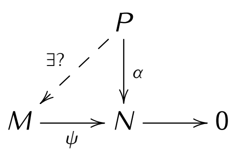
Proof:

(a) Choose a set $\{x_i\}_{i \in I}$ of generators of M (take all elements of M if necessary). Then define

$$\begin{aligned} \varphi : \bigoplus_{i \in I} R &\longrightarrow M \\ (r_i)_{i \in I} &\longmapsto \sum_{i \in I} r_i x_i . \end{aligned}$$

It follows that $M \cong \left(\bigoplus_{i \in I} R\right) / \ker \varphi$.

(b) We know that $\mathrm{Hom}(P, -)$ is left exact for any R-module P. It remains to prove that if $\psi : M \longrightarrow N$ is a surjective R-linear maps, then $\psi_* : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N) : \beta \longrightarrow \psi_*(\beta) = \psi \circ \beta$ is also surjective. So let $\alpha \in \mathrm{Hom}_R(P, N)$. We have the following situation:



Let $\{e_i\}_{i \in I}$ be an R-basis of P. Each $\alpha(e_i) \in N$ is in the image of ψ, so that for each $i \in I$ there exists $m_i \in M$ such that $\psi(m_i) = \alpha(e_i)$. Hence, there is a map $\beta : \{e_i\}_{i \in I} \longrightarrow M, e_i \mapsto m_i$. By the universal property of free modules this induces an R-linear map $\tilde{\beta} : P \longrightarrow M$ such that $\tilde{\beta}(e_i) = m_i$ $\forall\ i \in I$. Thus

$$\psi \circ \tilde{\beta}(e_i) = \psi(m_i) = \alpha(e_i) ,$$

so that $\psi \circ \tilde{\beta}$ and α coincide on the basis $\{e_i\}_{i \in I}$. By the uniqueness of $\tilde{\beta}$, we must have $\alpha = \psi \circ \tilde{\beta} = \psi_*(\tilde{\beta})$. ■

Injective modules

Proposition-Definition 6.5 (*Injective module*)

Let I be an R-module. Then the following are equivalent:

(a) The functor $\mathrm{Hom}_R(-, I)$ is exact.

(b) If $\varphi \in \mathrm{Hom}_R(L, M)$ is a injective morphism, then $\varphi_* : \mathrm{Hom}_R(M, I) \longrightarrow \mathrm{Hom}_R(L, I)$ is surjective (hence, any R-linear map $\alpha : L \longrightarrow I$ can be lifted to an R-linear map $\beta : M \longrightarrow I$, i.e., $\beta \circ \varphi = \alpha$).

(c) If $\eta : I \longrightarrow M$ is an injective R-linear map, then η splits, i.e., there exists $\rho : M \longrightarrow I$ such that $\rho \circ \eta = \mathrm{Id}_I$.

If I satisfies these equivalent conditions, then I is called **injective**.

Proof: Exercise. ■

Remark 6.6

Note that Condition (b) is particularly interesting when $L \leqslant M$ and φ is the inclusion.

Projective modules

Proposition-Definition 6.7 (*Projective module*)

Let P be an R-module. Then the following are equivalent:

(a) The functor $\mathrm{Hom}_R(P, -)$ is exact.

(b) If $\psi \in \mathrm{Hom}_R(M, N)$ is a surjective morphism of R-modules, then the morphism of abelian groups $\psi_* : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N)$ is surjective.

(c) If $\pi : M \longrightarrow P$ is a surjective R-linear map, then π splits, i.e., there exists $\sigma : P \longrightarrow M$ such that $\pi \circ \sigma = \mathrm{Id}_P$.

(d) P is isomorphic to a direct summand of a free R-module.

If P satisfies these equivalent conditions, then P is called **projective**.

Example 7

(a) If $R = \mathbb{Z}$, then every submodule of a free $\mathbb{Z}$-module is again free (main theorem on $\mathbb{Z}$-modules).

(b) Let e be an idempotent in R, that is $e^2 = e$. Then, $R \cong Re \oplus R(1-e)$ and Re is projective but not free if $e \neq 0, 1$.

(c) A product of modules $\{I_j\}_{j \in J}$ is injective if and only if each I_j is injective.

(d) A direct sum of modules $\{P_i\}_{i \in I}$ is projective if and only if each P_i is projective.

7 Tensor Products

Definition 7.1 (*Tensor product of R-modules*)

Let M be a right R-module and let N be a left R-module. Let F be the free abelian group (= free $\mathbb{Z}$-module) with basis $M \times N$. Let G be the subgroup of F generated by all the elements

$$\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N, \\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and} \\
&(mr, n) - (m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R.
\end{aligned}$$

The **tensor product of M and N (balanced over R)**, is the abelian group $M \otimes_R N := F/_G$. The class of $(m, n) \in F$ in $M \otimes_R N$ is denoted by $m \otimes n$.

Remark 7.2

(a) $M \otimes_R N = \langle m \otimes n \mid m \in M, n \in N \rangle_{\mathbb{Z}}$.

(b) In $M \otimes_R N$, we have the relations

$$\begin{aligned}
&(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \quad \forall m_1, m_2 \in M, \forall n \in N, \\
&m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \quad \forall m \in M, \forall n_1, n_2 \in N, \text{ and} \\
&mr \otimes n = m \otimes rn, \quad \forall m \in M, \forall n \in N, \forall r \in R.
\end{aligned}$$

In particular, $m \otimes 0 = 0 = 0 \otimes n \ \forall\ m \in M, \forall\ n \in N$ and $(-m) \otimes n = -(m \otimes n) = m \otimes (-n)$ $\forall\ m \in M, \forall\ n \in N$.

Definition 7.3 (*R-balanced map*)

Let M and N be as above and let A be an abelian group. A map $f : M \times N \longrightarrow A$ is called **R-balanced** if

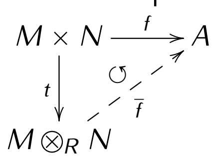
$$\begin{aligned}
&f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n), \quad \forall m_1, m_2 \in M, \forall n \in N, \\
&f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2), \quad \forall m \in M, \forall n_1, n_2 \in N, \\
&f(mr, n) = f(m, rn), \quad \forall m \in M, \forall n \in N, \forall r \in R.
\end{aligned}$$

Remark 7.4

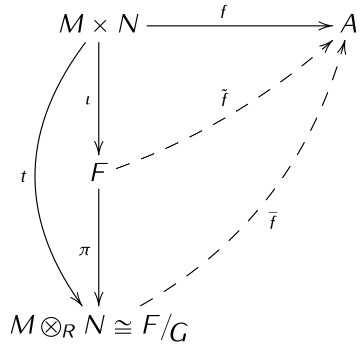
The canonical map $t : M \times N \longrightarrow M \otimes_R N, (m, n) \mapsto m \otimes n$ is R-balanced.

Proposition 7.5 (*Universal property of the tensor product*)

Let M be a right R-module and let N be a left R-module. For every abelian group A and every R-balanced map $f : M \times N \longrightarrow A$ there exists a unique $\mathbb{Z}$-linear map $\overline{f} : M \otimes_R N \longrightarrow A$ such that the following diagram commutes:



Proof: Let $\iota : M \times N \longrightarrow F$ denote the canonical inclusion, and let $\pi : F \longrightarrow F/G$ denote the canonical projection. By the universal property of the free $\mathbb{Z}$-module, there exists a unique $\mathbb{Z}$-linear map $\tilde{f} : F \longrightarrow A$ such that $\tilde{f} \circ \iota = f$. Since f is R-balanced, we have that $G \subseteq \ker(\tilde{f})$. Therefore, the universal property of the quotient yields the existence of a unique homomorphism of abelian groups $\overline{f} : F/G \longrightarrow A$ such that $\overline{f} \circ \pi = \tilde{f}$:



Clearly $t = \pi \circ \iota$, and hence $\overline{f} \circ t = \overline{f} \circ \pi \circ \iota = \tilde{f} \circ \iota = f$. ■

Remark 7.6

(a) Let $\{M_i\}_{i\in I}$ be a collection of right R-modules, M be a right R-module, N be a left R-module and $\{N_j\}_{i\in J}$ be a collection of left R-modules. Then, we have

$$\bigoplus_{i\in I} M_i \otimes_R N \cong \bigoplus_{i\in I}(M_i \otimes_R N)$$
$$M \otimes_R \bigoplus_{j\in J} N_j \cong \bigoplus_{j\in J}(M \otimes_R N_j).$$

(b) For every R-module M, we have $R \otimes_R M \cong M$ via $r \otimes m \mapsto rm$.

(c) If P be a free left R-module with basis X, then $M \otimes_R P \cong \bigoplus_{x\in X} M$.

(d) Let Q be a ring. Let M be a (Q, R)-bimodule and let N be an (R, S)-module. Then $M \otimes_R N$ can be endowed with the structure of a (Q, S)-bimodule via

$$q(m \otimes n)s = qm \otimes ns, \quad \forall q \in Q, \forall s \in S, \forall m \in M, \forall n \in N.$$

(e) If R is commutative, then any R-module can be viewed as an (R, R)-bimodule. Then, in particular, $M \otimes_R N$ becomes an R-module.

(f) **Tensor product of morphisms:** Let $f : M \longrightarrow M'$ be a morphism of right R-modules and $g : N \longrightarrow N'$ be a morphism of left R-modules. Then, by the universal property of the tensor product, there exists a unique $\mathbb{Z}$-linear map $f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

Proposition 7.7 (*Right exactness of the tensor product*)

(a) Let N be a left R-module. Then $- \otimes_R N : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$ is a right exact covariant functor.

(b) Let M be a right R-module. Then $M \otimes_R - : {}_R\mathbf{Mod} \longrightarrow \mathbf{Ab}$ is a right exact covariant functor.

Remark 7.8

The functors $-\otimes_R N$ and $M\otimes_R -$ are not left exact in general.

Definition 7.9 (*Flat module*)

A left R-module N is called **flat** if the functor $-\otimes_R N : \mathbf{Mod}_R \longrightarrow \mathbf{Ab}$ is a left exact functor.

Proposition 7.10

Any projective R-module is flat.

Proof: To begin with, we note that a direct sum of modules is flat if and only if each module in the sum is flat. Next, consider the free R-module $P = \bigoplus_{x\in X} Rx$. If

$$0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0$$

is a short exact sequence of right R-modules, then we obtain

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1\otimes_R\left(\bigoplus_{x\in X} R\right) & \xrightarrow{\varphi\otimes \mathrm{Id}_P} & M_2\otimes_R\left(\bigoplus_{x\in X} R\right) & \xrightarrow{\psi\otimes \mathrm{Id}_P} & M_3\otimes_R\left(\bigoplus_{x\in X} R\right) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \bigoplus_{x\in X} M_1 & \xrightarrow{(\varphi)_{x\in X}} & \bigoplus_{x\in X} M_2 & \xrightarrow{(\psi)_{x\in X}} & \bigoplus_{x\in X} M_3 & \longrightarrow & 0.
\end{array}$$

Since the original sequence is exact, so is the bottom sequence, and therefore so is the top sequence. Hence, $-\otimes_R P$ is exact when P is free.
Now, if N is a projective R-module, then $N\oplus N' = P'$ for some free R-module P' and for some R-module N'. It follows that N is flat, by the initial remark. ■