

TRIVIAL SOURCE ENDO-TRIVIAL MODULES FOR FINITE GROUPS WITH SEMI-DIHEDRAL SYLOW 2-SUBGROUPS

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Dedicated to Jon Carlson on the occasion of his 80th Birthday and to Jacques Thévenaz on the occasion of his 70th Birthday.

ABSTRACT. We finish off the classification of the endo-trivial modules of finite groups with Sylow 2-subgroups isomorphic to a semi-dihedral 2-group started by Carlson, Mazza and Thévenaz in their article *Endotrivial modules over groups with quaternion or semi-dihedral Sylow 2-subgroup* published in 2013.

1. INTRODUCTION

Endo-trivial modules play an important role in the representation theory of finite groups. For instance in the description of different types of equivalences between block algebras. These modules were introduced by E. C. Dade in 1978 [Dad78a, Dad78b]. They have been intensively studied since the beginning of the century and were classified in a number of special cases: e.g. for groups with cyclic, generalised quaternion, Klein-four or dihedral Sylow subgroups, for p -soluble groups, for the symmetric and the alternating groups and their Schur covers, for the sporadic groups their Schur covers, or for some infinite families of finite groups of Lie type. We refer the reader to the recent survey book [Maz19] by N. Mazza and the references therein for a complete introduction to this theory.

An *endo-trivial* module over the group algebra kG of a finite group G over an algebraically closed field k of prime characteristic p is by definition a finitely generated kG -module whose k -endomorphism algebra is isomorphic to the trivial module in the stable module category. The set of isomorphism classes of indecomposable endo-trivial kG -modules form an abelian group under the tensor product \otimes_k , which is denoted $T(G)$, and this group is known to be finitely generated. One of the central questions in this theory is to understand the structure of the group $T(G)$, and, in particular, of its torsion subgroup $TT(G)$.

Now, letting $X(G)$ be the subgroup of $TT(G)$ consisting of all one-dimensional kG -modules and $K(G)$ be the subgroup of $T(G)$ consisting of all the indecomposable endo-trivial kG -modules which are at the same time trivial source kG -modules, we have

$$\mathrm{Hom}(G, k^\times) \cong X(G) \subseteq K(G) \subseteq TT(G)$$

Date: February 14, 2021.

2010 Mathematics Subject Classification. Primary: 20C20. Secondary: 20C25, 20C33, 20C34, 20J05.

Key words and phrases. Endo-trivial modules, semi-dihedral 2-groups, Schur multiplier, trivial source modules, p -permutation modules, special linear and unitary groups.

The first author was partially supported by the Japan Society for Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C) 19K03416, 2019–2021. The second author acknowledges financial support by DFG SFB/TRR 195. This piece of work is part of Project A18 thereof.

and $K(G) = TT(G)$ unless a Sylow subgroup is cyclic, generalised quaternion or semi-dihedral (see [Maz19, Chapter 5]). Although it often happens that $X(G) = K(G)$, in general $X(G) \leq K(G)$. Furthermore, we emphasise that the determination of the structure of the endo-trivial modules lying in $K(G) \setminus X(G)$ is a very hard problem, to which, up to date, no general solution is known. Most of the work that has been done in previous articles provides case by case solutions to the calculation of the abelian group structure of $K(G)$, but in the vast majority of the cases does not provide information about the structure of the modules in $K(G) \setminus X(G)$.

In [CMT13] Carlson-Mazza-Thévenaz essentially described the structure of the group $T(G)$ of endo-trivial modules for groups with a semi-dihedral Sylow 2-subgroup. However they left open the question of computing the trivial source endo-trivial modules, i.e. the structure of the subgroup $K(G)$ of $T(G)$. The purpose of the present article is to finish off the determination of $K(G)$ in this case. In order to reach this aim we use three main ingredients, two of which were not available when [CMT13] was published:

1. The first one is a method we developed in [KL16] in order to treat finite groups with dihedral Sylow 2-subgroups, extended in [LT17a] to a more general method to relate the structure of $T(G)$ to that of $T(G/O_{p'}(G))$, which allows us to reduce the problem to groups with $O_{2'}(G) = 1$.
2. The second one is the classification of finite groups with semi-dihedral Sylow 2-subgroups modulo $O_{2'}(G)$ due to Alperin-Brauer-Gorenstein [ABG70].
3. The third main ingredient relies on major new results obtained by J. Grodal through the use of homotopy theory in [Gro18], or more precisely on a slight extension of the main theorem of [Gro18] recently obtained by D. Craven in [Cra20] and which provides us with purely group-theoretic techniques to deal with Grodal's description of $K(G)$ in [Gro18, Theorem 4.27]. The latter results will in particular enable us to treat families of groups related to the finite groups of Lie type $\mathrm{SL}_3(q)$ with $q \equiv 3 \pmod{4}$ and $\mathrm{SU}_3(q)$ with $q \equiv 1 \pmod{4}$.

With these tools, our main result is a description of the structure of the group of endo-trivial modules for groups with a semi-dihedral Sylow 2-subgroup as follows:

Theorem 1.1. *Let k be an algebraically closed field of characteristic 2 and let G be a finite group with a semi-dihedral Sylow 2-subgroup of order 2^m with $m \geq 4$. Then the following assertions hold.*

- (a) *If $G/O_{2'}(G) \not\cong \mathrm{PGL}_2^*(9)$, then $T(G) \cong X(G) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.*
- (b) *If $G/O_{2'}(G) \cong \mathrm{PGL}_2^*(9)$, then $T(G) \cong K(G) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ where*

$$K(G)/X(G) \leq \mathbb{Z}/3\mathbb{Z}.$$

Moreover, $K(G)/X(G) = 1$ if $G = \mathrm{PGL}_2^(9)$ and the bound $K(G)/X(G) \cong \mathbb{Z}/3\mathbb{Z}$ is obtained by the group $G = 3.\mathrm{PGL}_2^*(9)$.*

To explain the notation, there are exactly three groups H with $\mathrm{PSL}_2(9) < H < \mathrm{P}\Gamma\mathrm{L}_2(9)$ and $|H : \mathrm{PSL}_2(9)| = 2$, but amongst them only one has a semi-dihedral Sylow 2-subgroup and it is denoted $\mathrm{PGL}_2^*(9)$ (see Section 2). We also point out that in both cases the summand \mathbb{Z} is generated by the first syzygy module of the trivial kG -module and the summand $\mathbb{Z}/2\mathbb{Z}$ is generated by a torsion endo-trivial module which is explicitly determined in [CMT13, Proposition 6.4]. We note for completeness that the latter module had in fact already been constructed via different methods by Okuyama and Kawata, see [Kaw93, Theorem 5.1].

The paper is built up as follows. In Section 2 we introduce our notation. In Section 3 we quote main results on endo-trivial modules which we will use to prove Theorem 1.1. In Section 4 we state and prove preliminary results on groups with semi-dihedral Sylow 2-subgroups. In Section 5 we compute the trivial source endo-trivial modules for the special linear and special unitary groups and finally in Section 6 we prove Theorem 1.1.

2. NOTATION AND DEFINITIONS

Throughout this article, unless otherwise specified we adopt the following notation and conventions. All groups considered are assumed to be finite and all modules over finite group algebras are assumed to be finitely generated left modules. We let k denote an algebraically closed field of prime characteristic p , we let G denote a finite group of order divisible by p , and we let P be a Sylow p -subgroup of G .

We denote by SD_{2^m} the semi-dihedral group of order 2^m with $m \geq 4$, by C_a the cyclic group of order $a \geq 1$, by \mathfrak{A}_a and \mathfrak{S}_a the alternating and the symmetric groups on a letters, and we refer to the ATLAS [CCN⁺85] and [ABG70] for the definitions of further standard finite groups that occur in the statements of our main results. For a prime power q and $\varepsilon \in \{-1, 1\}$, we adopt the following notation. We set $\text{PSL}_n^\varepsilon(q) = \text{PSL}_n(q)$ (resp. $\text{GL}_n^\varepsilon(q) = \text{GL}_n(q)$, $\text{SL}_n^\varepsilon(q) = \text{SL}_n(q)$) if $\varepsilon = 1$ and $\text{PSL}_3^\varepsilon(q) = \text{PSU}_n(q)$ (resp. $\text{GU}_n^\varepsilon(q) = \text{GU}_n(q)$, $\text{SU}_n^\varepsilon(q) = \text{SU}_n(q)$) if $\varepsilon = -1$. We note that we consider $\text{GU}_n(q)$ as defined over \mathbb{F}_{q^2} . Moreover, we set $\text{SL}(2, q, \pm 1) := \{A \in \text{GL}_2(q) \mid \det(A) = \pm 1\}$ and $\text{SU}(2, q, \pm) := \{A \in \text{GU}_2(q) \mid \det(A) = \pm 1\}$. (These groups are denoted $\text{SL}_2^\pm(q)$ and $\text{SU}_2^\pm(q)$ in [ABG70], but we slightly alter this notation to avoid confusion.) Also, if $q = r^{2f}$ is a positive even power of an odd prime number r , then there are exactly three groups H with $\text{PSL}_2(q) < H < \text{P}\Gamma\text{L}_2(q)$ and $|H : \text{PSL}_2(q)| = 2$. One is $\text{PGL}_2(q)$, one is contained in $\text{PSL}_2(q) \rtimes \langle F \rangle$ where F is the Frobenius automorphism on \mathbb{F}_q , and the third one is denoted $\text{PGL}_2^*(q)$ (see [Gor69, p.335]). In this article, we will only work with $\text{PGL}_2^*(q)$ and we will essentially need the fact that $\text{PSL}_2(q)$ is normal in $\text{PGL}_2^*(q)$. Hence we will write $\text{PGL}_2^*(q) = \text{PSL}_2(q).2$ without making the difference with the other two extensions. For $\emptyset \neq S \subseteq G \ni g, x$ we write ${}^gS := \{gsg^{-1} \mid s \in S\}$ and $x^g := g^{-1}xg$. We let $O_{p'}(G)$, resp. $O_p(G)$, be the largest normal p' -subgroup, resp. p -subgroup, of G and $O^{p'}(G)$ the smallest normal subgroup of G whose quotient is a p' -group. Following [Cra20, Section 2] we define K_G° to be the normal subgroup of $N_G(P)$ generated by $N_G(P) \cap O^{p'}(N_G(Q))$ for all $N_G(P)$ -conjugacy classes of subgroups $1 < Q \leq P$. Clearly $K_G^\circ \trianglelefteq N_G(P)$.

We will see the Schur multiplier of G as $H^2(G, \mathbb{C}^\times) =: M(G)$ and we recall that it is well-known that $H^2(G, k^\times) \cong M(G)_{2'}$ (see e.g. [Kar87, Proposition 2.1.14]). We recall that a p' -representation group of G (or a representation group of G relative to k) is a p' -central extension \tilde{G} of G of minimal order with the projective lifting property. We emphasise that the kernel of such a p' -central extension is isomorphic to $H^2(G, k^\times)$, and if the Schur multiplier is a p' -group, then a p' -representation group is just a representation group in the usual sense. For further details on this notion we refer the reader to the expository note [LT17b].

If M is a kG -module, then we denote by M^* the k -dual of M and by $\text{End}_k(M)$ its k -endomorphism algebra, both of which are endowed with the kG -module structure given by the conjugation action of G . We recall that a kG -module M is called *endo-trivial* if

there is an isomorphism of kG -modules

$$\mathrm{End}_k(M) \cong k \oplus (\mathrm{proj})$$

where k denotes the trivial kG -module and (proj) denotes a projective kG -module or possibly the zero module. Any endo-trivial kG -module M decomposes as a direct sum $M = M_0 \oplus (\mathrm{proj})$ where M_0 , the projective-free part of M , is indecomposable and endo-trivial. The set $T(G)$ of all isomorphism classes of indecomposable endo-trivial kG -modules endowed with the composition law $[M] + [L] := [(M \otimes_k L)_0]$ is an abelian group called the *group of endo-trivial modules of G* . The zero element is the class $[k]$ of the trivial module and $-[M] = [M^*]$. By a result of Puig for finite p -groups, extended to arbitrary finite groups by Carlson-Mazza-Nakano, the group $T(G)$ is known to be a finitely generated abelian group (see [Maz19, Theorem 2.3]).

We let $X(G)$ denote the group of one-dimensional kG -modules endowed with the tensor product \otimes_k . Clearly $X(G) \leq T(G)$ and we recall that

$$X(G) \cong \mathrm{Hom}(G, k^\times) \cong (G/[G, G])_{p'}.$$

In particular, it follows that $X(G) = \{[k]\}$ when G is p' -perfect. Furthermore, we let $K(G)$ denote the subgroup of $T(G)$ consisting of the isomorphism classes of indecomposable endo-trivial kG -modules with a trivial source. It follows easily from the theory of vertices and sources that $K(G)$ is precisely the kernel of the restriction homomorphism

$$\mathrm{Res}_P^G : T(G) \longrightarrow T(P), [M] \mapsto [\mathrm{Res}_P^G(M)_0].$$

Clearly $X(G) \leq K(G)$ since the restriction of a one-dimensional kG -module to P is the trivial module. Moreover, $K(G)$ is a finite group as it is made up of isomorphism classes of trivial source kG -modules, hence $K(G) \leq TT(G)$. Moreover, by the main result of [CT00], we have $K(G) = TT(G)$ unless P is cyclic, generalised quaternion, or semi-dihedral.

3. KNOWN RESULTS

To begin with, we quickly review the results about $T(G)$ in the semi-dihedral case obtained by Carlson-Thévenaz in [CT00] and Carlson-Mazza-Thévenaz in [CMT13].

Theorem 3.1. *Let $p = 2$ and let G be a finite group with a Sylow 2-subgroup $P \cong \mathrm{SD}_{2^m}$ of order 2^m with $m \geq 4$. Then the following assertions hold.*

- (a) [CT00, Theorem 7.1] $T(P) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.
- (b) [CMT13, Proposition 6.1] $T(G) \cong K(G) \oplus \mathrm{Im}(\mathrm{Res}_P^G)$.
- (c) [CMT13, Proposition 6.4] $\mathrm{Res}_P^G : T(G) \longrightarrow T(P)$ is a split surjective group homomorphism.
- (d) [CMT13, Proposition 6.4] $TT(G) \cong K(G) \oplus \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}/2\mathbb{Z}$ summand is generated by a self-dual torsion endo-trivial module which is not a trivial source module.
- (e) [CMT13, Corollary 6.5] If $P = N_G(P)$, then $K(G) = \ker(\mathrm{Res}_P^G) = \{[k]\}$ and hence $T(G) \cong T(P) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.

Open Question 3.2. *The remaining open question in the article [CMT13] by Carlson-Mazza-Thévenaz about finite groups G with semi-dihedral Sylow 2-subgroups is to compute the structure of the group $K(G)$ when the Sylow 2-subgroups are not self-normalising.*

Next, we state below the main results which we will use in our proof of Theorem 1.1. However, they are all valid in arbitrary prime characteristic p .

Lemma 3.3 ([MT07, Lemma 2.6]). *Let G be a finite group and let P be a Sylow p -subgroup of G . If ${}^xP \cap P \neq 1$ for every $x \in G$, then $K(G) = X(G)$. In particular, if $O_p(G) > 1$, then $K(G) = X(G)$.*

Then, the following lemma will be applied to semi-direct products of the form $G = N \rtimes H$ with $p \nmid |G : N|$.

Lemma 3.4 ([Maz19, Theorem 5.1.(4.)]). *Let G be a finite group and let $N \trianglelefteq G$ such that $p \nmid |G : N|$. If $K(N) = X(N)$, then $K(G) = X(G)$.*

We will also use the following result from our previous paper on endo-trivial modules for groups with dihedral Sylow 2-subgroups.

Theorem 3.5 ([KL16, Theorem 1.1]). *Let G be a finite group with p -rank at least 2 and which does not admit a strongly p -embedded subgroup. Let $H \trianglelefteq G$ be a normal subgroup such that $p \nmid |H|$. If $H^2(G, k^\times) = 1$, then*

$$K(G) = X(G) + \text{Inf}_{G/H}^G(K(G/H)).$$

In case $H^2(G, k^\times) \neq 1$, then we may apply the following generalisation of the above result.

Theorem 3.6 ([LT17a, Theorem 1.1]). *Let G be a finite group with p -rank at least 2 and which does not admit a strongly p -embedded subgroup. Let \tilde{Q} be any p' -representation group of the group $Q := G/O_{p'}(G)$.*

(a) *There exists an injective group homomorphism*

$$\Phi_{G, \tilde{Q}} : T(G)/X(G) \longrightarrow T(\tilde{Q})/X(\tilde{Q}).$$

In particular, $\Phi_{G, \tilde{Q}}$ maps the class of $\text{Inf}_Q^G(W)$ to the class of $\text{Inf}_{\tilde{Q}}^{\tilde{Q}}(W)$, for any endo-trivial kQ -module W .

(b) *The map $\Phi_{G, \tilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G, \tilde{Q}} : K(G)/X(G) \longrightarrow K(\tilde{Q})/X(\tilde{Q}).$$

(c) *In particular, if $K(\tilde{Q}) = X(\tilde{Q})$, then $K(G) = X(G)$.*

Finally, we will apply a recent result obtained by David Craven in [Cra20], which gives a purely group-theoretic method in order to use Grodal's homotopy-theoretical description of $K(G)$ in [Gro18, Theorem 4.27].

Lemma 3.7 ([Cra20, Section 2]). *Let G be a finite group and let P be a Sylow p -subgroup of G . If $K_G^\circ = N_G(P)$, then $K(G) \cong \{[k]\}$.*

Proof. Assuming $K_G^\circ = N_G(P)$, by [Cra20, Theorem 2.3 and the remark before Theorem 2.3] we have

$$K(G) \cong (N_G(P)/K_G^\circ)^{\text{ab}} = 1.$$

The claim follows. □

4. SOME PROPERTIES OF GROUPS WITH SEMI-DIHEDRAL SYLOW 2-SUBGROUPS

From this point forward and for the remainder of this article, we assume that the field k has characteristic $p = 2$.

The following result provides a classification of finite groups with semi-dihedral Sylow 2-subgroups, which was obtained by Benjamin Sambale in an unpublished note as a byproduct of the results of Alperin-Brauer-Gorenstein in [ABG70]. A proof can be found in [KLS20, Theorem 3.1].

Proposition 4.1 ([KLS20, Theorem 3.1]). *Let G be a finite group with a Sylow 2-subgroup isomorphic to SD_{2^m} ($m \geq 4$) and $O_{2'}(G) = 1$. Let r be a prime number and $q = r^n$ be a positive power of r . Then one of the following holds:*

- (SD1) $G \cong \text{SD}_{2^m}$;
- (SD2) $G \cong \text{M}_{11}$ and $m = 4$;
- (SD3) $G \cong \text{SL}(2, q, \pm 1) \rtimes C_d$ where $q \equiv -1 \pmod{4}$, $d \mid n$ is odd and $(q+1)_2 = 2^{m-2}$;
- (SD4) $G \cong \text{SU}(2, q, \pm 1) \rtimes C_d$ where $q \equiv 1 \pmod{4}$, $d \mid n$ is odd and $(q-1)_2 = 2^{m-2}$;
- (SD5) $G \cong \text{PGL}_2^*(q^2) \rtimes C_d$ where r is odd, $d \mid n$ is odd and $(q^2-1)_2 = 2^{m-1}$;
- (SD6) $G \cong \text{PSL}_3^\varepsilon(q).H$ where $q \equiv -\varepsilon \pmod{4}$, $H \leq C_{(3, q-\varepsilon)} \times C_n$ has odd order and $(q+\varepsilon)_2 = 2^{m-1}$.

In addition, two crucial results for the present article are given by the following lemma and proposition, which will allow us to apply Theorem 3.5 and Theorem 3.6.

Lemma 4.2. *A finite group with a semi-dihedral Sylow 2-subgroup does not admit any strongly 2-embedded subgroup.*

Proof. The Bender-Suzuki theorem [Ben71, Satz 1] states that a finite group G with a strongly 2-embedded subgroup H has one of the following forms:

1. G has cyclic or generalised quaternion Sylow 2-subgroups and H contains the centraliser of an involution; or
2. $G/O_{2'}(G)$ has a normal subgroup of odd index isomorphic to one of the simple groups $\text{PSL}_2(q)$, $\text{Sz}(q)$ or $\text{PSU}_3(q)$ where $q \geq 4$ is a power of 2 and H is $O_{2'}(G)N_G(P)$ for a Sylow 2-subgroup P of G .

Therefore, it follows from Proposition 4.1 that such a group cannot admit a semi-dihedral Sylow 2-subgroup. \square

Proposition 4.3. *Let G be a finite group with a semi-dihedral Sylow 2-subgroup $P \cong \text{SD}_{2^m}$ for some $m \geq 4$ and $O_{2'}(G) = 1$. Let r be a prime number and $q = r^n$ be a positive power of r . Then the following assertions hold.*

- (a) *If $G = \text{SD}_{2^m}$, then $H^2(G, k^\times) = 1$.*
- (b) *If $G = \text{M}_{11}$, then $H^2(G, k^\times) = 1$.*
- (c) *If $G = \text{SL}(2, q, \pm 1) \rtimes C_d$ where $q \equiv -1 \pmod{4}$ and $d \mid n$ is odd, then $H^2(G, k^\times) = 1$.*
- (d) *If $G = \text{SU}(2, q, \pm 1) \rtimes C_d$ where $q \equiv 1 \pmod{4}$ and $d \mid n$ is odd, then $H^2(G, k^\times) = 1$, unless $G = \text{SU}(2, 9, \pm 1)$, in which case $H^2(G, k^\times) \cong C_3$.*
- (e) *If $G = \text{PGL}_2^*(q^2) \rtimes C_d$ where q^2 is odd and $d \mid n$ is odd, then $H^2(G, k^\times) = 1$, unless $G = \text{PGL}_2^*(9)$, in which case $H^2(G, k^\times) \cong C_3$.*
- (f) *If $G = \text{PSL}_3^\varepsilon(q).H$ where $q \equiv -\varepsilon \pmod{4}$ and $H \leq C_{(3, q-\varepsilon)} \times C_n$ has odd order, then $|H^2(G, k^\times)| = (3, q-\varepsilon)$ if H is cyclic, and $|H^2(G, k^\times)| = 9$ else.*

Proof. Set $h := |H^2(G, k^\times)|$. In order to compute h we recall that $H^2(G, k^\times) \cong M(G)_{2'}$ (see [Kar87, Proposition 2.1.14]) and if $N \trianglelefteq G$ such that G/N is cyclic then, by [Jon74, Theorem 3.1(i)], we have

$$|M(G)| \mid |M(N)| \cdot |N/[N, N]|. \quad (*)$$

We compute:

- (a) Because the Schur multiplier of a cyclic group is trivial, we obtain from [Kar87, Theorem 2.1.2(i)] (or [Isa08, Corollary 5.4]) that if p is an odd prime divisor of $|M(G)|$ then a Sylow p -subgroup of G must be noncyclic, hence $|M(G)_{2'}| = 1$.
- (b) See the ATLAS [CCN⁺85, p.18].
- (c) First, we have $\mathrm{SL}(2, q, \pm 1) \cong \mathrm{SL}_2(q).2$ (see [ABG70, Chapter I, p.4]). Thus (*) yields

$$M(\mathrm{SL}(2, q, \pm 1)) = 1$$

since $\mathrm{SL}_2(q)$ is perfect and has a trivial Schur multiplier as $q \equiv 3 \pmod{4}$ (see [Kar87, 7.1.1.Theorem]). Therefore, we may apply (*) again to $G = N \rtimes C_d$ with $N = \mathrm{SL}(2, q, \pm 1)$. Because $|N/[N, N]| = 2$ we obtain

$$|M(G)| \mid |M(N)| \cdot |N/[N, N]| = 2$$

and it follows that $h = |M(G)_{2'}| = 1$.

- (d) We have $\mathrm{SU}(2, q, \pm 1) \cong \mathrm{SU}_2(q).2$ (see [ABG70, p.4]) and $\mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$. Therefore, if $q \neq 9$ we have $M(\mathrm{SU}_2(q)) \cong M(\mathrm{SL}_2(q)) = 1$ (see [Kar87, 7.1.1.Theorem]) and the claim follows by the same argument as in (c), applying (*) twice.

Next, if $q = 9$, then we first observe that necessarily $d = 1$. Moreover, we claim that $\mathrm{SU}(2, 9, \pm 1) \cong 2.\mathrm{PGL}_2(9)$. Indeed, as $\mathrm{PSL}_2(9) \cong \mathfrak{A}_6$, by [ABG70, diagram on p.4], $Q := \mathrm{SU}(2, 9, \pm 1) \cong C_2.\mathrm{PSL}_2(9).C_2 \cong C_2.\mathfrak{A}_6.C_2$. Hence, we read from the ATLAS [CCN⁺85, p.4] that

$$Q \in \{2.\mathfrak{A}_6.2_1 = 2.\mathfrak{S}_6, 2.\mathfrak{A}_6.2_2 = 2.\mathrm{PGL}_2(9), 2.\mathfrak{A}_6.2_3 = 2.\mathrm{PGL}_2^*(9)\}.$$

(Here we use the ATLAS notation.) Considering the quotient $\bar{Q} := Q/Z(Q) = Q/C_2$ and letting $\bar{P} \in \mathrm{Syl}_2(\bar{Q})$, we have $\bar{P} \cong D_8 \times C_2$ if $Q \cong 2.\mathfrak{S}_6$, $\bar{P} \cong D_{16}$ if $Q \cong 2.\mathrm{PGL}_2(9)$, and $\bar{P} \cong \mathrm{SD}_{16}$ if $Q \cong 2.\mathrm{PGL}_2^*(9)$. However, as a Sylow 2-subgroup of Q is semi-dihedral of order 32 and $\mathrm{SD}_{32}/Z(\mathrm{SD}_{32}) \cong D_{16}$, we must have $Q \cong 2.\mathrm{PGL}_2(9)$. Hence, by [Kar87, 2.1.15 Corollary] and the ATLAS [CCN⁺85, p.4], we obtain that

$$M(G)_{2'} \cong C_3.$$

- (e) We have $\mathrm{PGL}_2^*(q^2) = \mathrm{PSL}_2(q^2).2$ (see [Gor69, p.335]). Now, if $q^2 \neq 9$, then $\mathrm{PSL}_2(q^2)$ is perfect and has a Schur multiplier of order 2. Therefore applying (*) twice as in (c) it follows that $h = |M(G)_{2'}| = 1$. If $q^2 = 9$, then necessarily $d = 1$, and because $\mathrm{PGL}_2^*(9) \cong \mathfrak{A}_6.2$ we read from the ATLAS [CCN⁺85, p.4] that $h = 3$.
- (f) Write $G = N.H$ with $N := \mathrm{PSL}_3^\varepsilon(q)$ ($q \equiv -\varepsilon \pmod{4}$). First assume that H is cyclic. Because N is perfect, by (*), we have that

$$|M(G)| \mid |M(N)| = (3, q - \varepsilon)$$

as $M(N) \cong C_{(3,q-\varepsilon)}$ (see e.g. [CCN⁺85, Chapter 2]). Hence $h = |M(G)_{2'}| |(3, q-\varepsilon)|$. Next, if H is not cyclic, we have $H \cong C_3 \times C_a$ with $3 \mid a$. For $X := N.C_3$ we obtain that $|M(X)| |M(N)| = 3$ by (*). Then applying (*) a second time, we get

$$|M(G)| \mid |M(X)| \cdot |X/[X, X]| = 9.$$

Hence $h = |M(G)_{2'}| |9|$.

□

Remark 4.4. We note that if in case (SD6) of Proposition 4.1 the extension $G = \text{PSL}_3^\varepsilon(q).H$ is split, then it follows from a general result of K. Tahara [Tah72, Theorem 2] on the second cohomology groups of semi-direct products that $H^2(G, k^\times) \cong C_{(3,q-\varepsilon)}$ if H is cyclic and $H^2(G, k^\times) \cong C_3 \times C_3$ if H is non-cyclic.

5. ENDOTRIVIAL MODULES FOR $\text{SL}_3^\varepsilon(q)$ WITH $q \equiv -\varepsilon \pmod{4}$

In this section, we compute $K(G)$ for $G = \text{SL}_3^\varepsilon(q)$ with $q \equiv -\varepsilon \pmod{4}$ in characteristic 2. We note that for the special linear group $G = \text{SL}_3(q)$ with $q \equiv -1 \pmod{4}$ the structure of $K(G)$ is given in [CMN16, Theorem 9.2(b)(ii)], namely $K(G) = X(G)$. However, the proof given to this fact in [CMN16] contains an error: the authors assumed that a Sylow 2-subgroup S of the simple group $H = \text{PSL}_3(q)$ is always self-normalising, which is not correct in general, as $N_H(S) \cong S \times Z$ where Z is a cyclic group of order $(q-1)_{2'}/(q-1, 3)$. (See e.g. [Kon05, Corollary on p.2]). More precisely, identifying $\text{GL}_2(q)$ with the subgroup of $\text{SL}_3(q)$ made up of the 2×2 left upper block matrices as below, we have $Z = O_{2'}(Z(\text{GL}_2(q)))/U(\text{SL}_3(q))$. For this reason, we treat both $\text{SL}_3(q)$ and $\text{SU}_3(q)$.

Throughout this section we let $G := \text{SL}_3^\varepsilon(q)$ and $\tilde{G} := \text{GL}_3^\varepsilon(q)$ with $q \equiv -\varepsilon \pmod{4}$ an odd prime power and we define \bar{q} to be q if $\varepsilon = 1$ and q^2 if $\varepsilon = -1$. Furthermore, we let

$$\iota : \text{GL}_2^\varepsilon(q) \longrightarrow \text{SL}_3^\varepsilon(q), a \mapsto \begin{pmatrix} a & 0 \\ 0 & \det(a)^{-1} \end{pmatrix}$$

be the natural embedding. Now, in order to describe the normaliser of $P \in \text{Syl}_2(G)$, we follow the procedure described in [SFT18, Sections 7 and 8] to obtain $N_G(P)$ from the normaliser $N_{\tilde{G}}(\tilde{P})$ of a Sylow 2-subgroup \tilde{P} of \tilde{G} as given by Carter-Fong [CF64]. Firstly, as the 2-adic expansion of 3 is $3 = 2^{r_1} + 2^{r_2}$ with $r_1 = 1$ and $r_2 = 0$ we have $\tilde{P} \cong \prod_{i=1}^2 S_{r_i}^\varepsilon(q)$ where $S_{r_i}^\varepsilon(q) \in \text{Syl}_2(\text{GL}_{2^{r_i}}^\varepsilon(q))$ and

$$N_{\tilde{G}}(\tilde{P}) \cong \tilde{P} \times C_{(q-\varepsilon)_{2'}} \times C_{(q-\varepsilon)_{2'}}.$$

Concretely, we may assume that \tilde{P} is realised by embedding $\prod_{i=1}^2 S_{r_i}^\varepsilon(q) \leq \prod_{i=1}^2 \text{GL}_{2^{r_i}}^\varepsilon(q)$ block-diagonally in a natural way. Moreover, for $1 \leq j \leq 2$ the corresponding factor $C_{(q-\varepsilon)_{2'}}$ is embedded as $O_{2'}(Z(\text{GL}_{2^{r_j}}^\varepsilon(q)))$, so that an arbitrary element of $N_{\tilde{G}}(\tilde{P})$ is of the form xz with $x \in \tilde{P}$ and z is a diagonal matrix of the form $z = \text{diag}(\lambda_1 I_2, \lambda_2)$ with $\lambda_1, \lambda_2 \in C_{(q-\varepsilon)_{2'}} \leq \mathbb{F}_{\bar{q}}^\times$ and I_2 the identity matrix in $\text{GL}_2(\bar{q})$.

Lemma 5.1. *The following assertions hold.*

- (a) $P := \tilde{P} \cap G = \{\iota(x) \mid x \in S_1^\varepsilon(q)\}$ is a Sylow 2-subgroup of G which is normal in \tilde{P} .
- (b) $N_G(P) = P \times O_{2'}(C_G(P))$ where

$$O_{2'}(C_G(P)) = \{\text{diag}(\lambda_1 I_2, \lambda_1^{-2}) \mid \lambda_1 \in C_{(q-\varepsilon)_{2'}} \leq \mathbb{F}_{\bar{q}}^\times\} \cong C_{(q-\varepsilon)_{2'}}.$$

(c) If $Q \leq G$ is such that $Q \cong C_2 \times C_2$, then $O^{2'}(N_G(Q)) = N_G(Q)$.

Proof. Part (a) is given by [SFT18, Sections §8.1]. For part (b), first, as P is semi-dihedral its automorphism group $\text{Aut}(P)$ is a 2-group and it follows that

$$N_G(P) = PC_G(P) = P \times Z \quad \text{with } Z := O_{2'}(C_G(P)).$$

Now by [SFT18, Sections §8.1], we have $N_G(P) = N_{\tilde{G}}(\tilde{P}) \cap G$ and the claim follows from the above description of $N_{\tilde{G}}(\tilde{P})$.

Part (c) is obtained as follows. Let ζ is a generator of the subgroup $C_{(q-\varepsilon)_{2'}} \leq \mathbb{F}_q^\times$. Consider the diagonal matrices

$$u := \text{diag}(1, -1, -1), v := \text{diag}(-1, 1, -1), x := \text{diag}(\zeta, \zeta, \zeta^{-2}), y := \text{diag}(\zeta, 1, \zeta^{-1}) \in G.$$

We can set $Q := \langle u, v \rangle \leq G$ since all subgroups of G isomorphic to $C_2 \times C_2$ are G -conjugate. Furthermore, consider the matrices

$$t := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad a := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in G$$

and set $H := O^{2'}(N_G(Q))$. Then we have

$$a^t = a^2uv, u^a = v, v^a = uv, (uv)^a = u, x^a = xy^{-3}, x^t = x^{-2}y^3, y^a = xy^{-2}, y^t = x^{-1}y^2.$$

Clearly $u, v, t \in H$ since H is generated by the 2-elements of $N_G(Q)$. Moreover $a \in H$, so that $x, y \in H$ as well. The assertion is proved. \square

Proposition 5.2. *If $G = \text{SL}_3^\varepsilon(q)$ with $q \equiv -\varepsilon \pmod{4}$ an odd prime power, then*

$$K(G) = X(G) = \{[k]\}.$$

Proof. Let P and Q be as in Lemma 5.1. Thanks to Lemma 3.7(b) it is enough to prove that $N_G(P) = K_G^\circ$. By Lemma 5.1(b), we have $N_G(P) = P \times O_{2'}(C_G(P))$, hence it is clear that $P = O^{2'}(N_G(P)) = N_G(P) \cap O^{2'}(N_G(P)) \leq K_G^\circ$. Now,

$$O_{2'}(C_G(P)) \leq C_G(P) \leq C_G(Q) \leq N_G(Q) = O^{2'}(N_G(Q))$$

where the last equality holds by Lemma 5.1(c). Thus,

$$O_{2'}(C_G(P)) \leq N_G(P) \cap O^{2'}(N_G(Q)) \leq K_G^\circ.$$

\square

6. PROOF OF THEOREM 1.1

Throughout this section we let G denote a finite group such that $P \in \text{Syl}_2(G)$ is semi-dihedral of order 2^m for some $m \geq 4$. We let r be a prime number and $q = r^n$ be a positive power of r . We write $Q := G/O_{2'}(G)$.

In order to prove Theorem 1.1, we go through the possibilities for Q given by Proposition 4.1. As the 2-rank of P is 2 and G does not have any strongly 2-embedded subgroup by Lemma 4.2, we may apply Theorem 3.5 and Theorem 3.6.

Proposition 6.1. *If $G/O_{2'}(G) \not\cong \text{PGL}_2^*(9)$, then $K(G) = X(G)$.*

Proof. To begin with, we assume that $H^2(Q, k^\times) = 1$. In this case it suffices to prove that $K(Q) = X(Q)$, since it then follows from Theorem 3.5 that $K(G) = X(G)$. We go through the possibilities for Q given by Proposition 4.1 and Proposition 4.3 as follows.

Case 1: $Q \cong P$. Then clearly $K(Q) = \{[k]\}$. (Although in this case G is 2-nilpotent and hence the fact that $K(G) = X(G)$ is proved in [CMT11, Theorem 5.1]).

Case 2: $Q \cong M_{11}$. As M_{11} has a self-normalising Sylow 2-subgroup it follows from Theorem 3.1(e) that $K(Q) = X(Q) = \{[k]\}$. (See [LM15, §4.1].)

Case 3: $Q \cong \mathrm{SL}(2, q, \pm 1) \rtimes C_d$ with $q \equiv -1 \pmod{4}$ and $d \mid n$ is odd. In this case set $N := \mathrm{SL}(2, q, \pm 1)$. Then $K(N) = X(N)$ by Lemma 3.3 because G admits a non-trivial central 2-subgroup $\{\pm I_2\}$. Therefore, Lemma 3.4 yields $K(Q) = X(Q)$, because $N \trianglelefteq Q$ with odd index.

Case 4: $Q \cong \mathrm{SU}(2, q, \pm 1) \rtimes C_d$ with $9 \neq q \equiv 1 \pmod{4}$ and $d \mid n$ is odd. In this case set $N := \mathrm{SU}(2, q, \pm 1)$. Then $K(N) = X(N)$ by Lemma 3.3 because G admits a non-trivial central 2-subgroup $\{\pm I_2\}$. Therefore, Lemma 3.4 yields $K(Q) = X(Q)$, because $N \trianglelefteq Q$ with odd index.

Case 5: $Q \cong \mathrm{PGL}_2^*(q^2) \rtimes C_d$ where $q^2 \neq 9$ is odd and d is an odd divisor of n . In this case set $N := \mathrm{PGL}_2^*(q^2) \cong \mathrm{PSL}_2(q^2).2$. As a Sylow 2-subgroup of $\mathrm{PSL}_2(q^2)$ is self-normalising, so is a Sylow 2-subgroup of N . Therefore $K(N) = X(N)$ by Theorem 3.1(e) and hence it follows from Lemma 3.4 that $K(Q) = X(Q)$.

Case 6: $Q \cong \mathrm{PSL}_3^\varepsilon(q).H$ where $q \equiv -\varepsilon \pmod{4}$ and $H \leq C_{(3, q-\varepsilon)} \times C_n$ has odd order and $H^2(Q, k^\times) = 1$. In this case set $N := \mathrm{PSL}_3^\varepsilon(q)$. As $K(\mathrm{SL}_3^\varepsilon(q)) = X(\mathrm{SL}_3^\varepsilon(q))$ by Proposition 5.2 and $\mathrm{SL}_3^\varepsilon(q)$ is a 2'-representation group of $\mathrm{PSL}_3^\varepsilon(q)$ we also have that $K(N) = X(N)$ by Theorem 3.6(c). Therefore $K(Q) = X(Q)$ by Lemma 3.4 since $N \trianglelefteq Q$ with odd index.

Next we assume that $H^2(G, k^\times) \neq 1$ and let \tilde{Q} be a 2'-representation group of Q . In this case it suffices to prove that $K(\tilde{Q}) = X(\tilde{Q})$, because it then follows from Theorem 3.6(c) that that $K(G) = X(G)$. We go through the possibilities for Q given by Proposition 4.1 and Proposition 4.3 as follows.

Case 7: $Q \cong \mathrm{SU}(2, 9, \pm 1)$. First, recall that $\mathrm{SU}(2, 9, \pm 1)$ is isomorphic to $2.\mathrm{PGL}_2(9)$. (See proof of Proposition 4.3(d)). By Proposition 4.3(d) we have $H^2(Q, k^\times) \cong C_3$ and we may choose \tilde{Q} to be $6.\mathrm{PGL}_2(9)$. Therefore Lemma 3.3 yields $K(\tilde{Q}) = X(\tilde{Q})$ because \tilde{Q} admits a normal (central) subgroup of order 2.

Case 8: $Q \cong \mathrm{PSL}_3^\varepsilon(q).H$ where $q \equiv -\varepsilon \pmod{4}$ and $H \leq C_{(3, q-\varepsilon)} \times C_n$ has odd order and $|H^2(Q, k^\times)| = 3$. Then we can take $\tilde{Q} = N.H$ with $N := \mathrm{SL}_3^\varepsilon(q)$. Now, $K(N) = X(N)$ by Proposition 5.2, so that $K(\tilde{Q}) = X(\tilde{Q})$ by Lemma 3.4 because N is normal of odd index in \tilde{Q} .

Case 9: $Q \cong \mathrm{PSL}_3^\varepsilon(q).H$ where $q \equiv -\varepsilon \pmod{4}$ and $H \leq C_{(3, q-\varepsilon)} \times C_n$ has odd order and $|H^2(Q, k^\times)| = 9$. In this case \tilde{Q} is a central extension of $Q_1 := \mathrm{SL}_3^\varepsilon(q).H$ with kernel $Z \cong C_3$. Since $\mathrm{SL}_3^\varepsilon(q)$ is normal in Q_1 , there exists a normal subgroup $Y \trianglelefteq \tilde{Q}$ containing Z and such that $Y/Z \cong \mathrm{SL}_3^\varepsilon(q)$ and as $\mathrm{SL}_3^\varepsilon(q)$ is its own 2'-representation group, we have that $Y \cong Z \times \mathrm{SL}_3^\varepsilon(q)$. Therefore, as $K(\mathrm{SL}_3^\varepsilon(q)) = X(\mathrm{SL}_3^\varepsilon(q))$ by Proposition 5.2, it follows from Lemma 3.4, applied a first time, that $K(Y) = X(Y)$ and applied a second time that $K(\tilde{Q}) = X(\tilde{Q})$. \square

Lemma 6.2.

- (a) If $G = \mathrm{PGL}_2^*(9)$, then $K(G) = X(G)$.
- (b) If $G = 3.\mathrm{PGL}_2^*(9)$, then $K(G) \cong \mathbb{Z}/3\mathbb{Z}$. Moreover, the indecomposable representatives of the two non-trivial elements of $K(G)$ lie in the two distinct faithful and dual 2-blocks with full defect B and B^* of G . Their Loewy and socle series are respectively

$$\begin{array}{|c|} \hline 9 \\ \hline 9 & 6 \\ \hline 9 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline 9^* \\ \hline 9^* & 6^* \\ \hline 9^* \\ \hline \end{array}$$

where 9 (resp. 6) denotes the unique 9-dimensional (resp. 6-dimensional) simple kB -module and 6^* (resp. 9^*) is the k -dual of 6 (resp. 9).

- (c) If $G/O_{2'}(G) \cong \mathrm{PGL}_2^*(9)$, then $K(G)/X(G)$ is isomorphic to a subgroup of $\mathbb{Z}/3\mathbb{Z}$.

Proof. (a) The group $\mathrm{PGL}_2^*(9)$ has a semi-dihedral Sylow 2-subgroup $P \cong \mathrm{SD}_{16}$, which is self-normalising. Hence the claim is immediate from Theorem 3.1(e).

- (b) We treat the group $G = 3.\mathrm{PGL}_2^*(9)$ entirely via computer algebra using MAGMA [BCP97].

First, we note that the block structure of G can be found in the *decomposition matrices* section at the Modular Atlas Homepage [WTP⁺, $A_6.2_3$], where we also read that B has exactly two simple modules, one of dimension 6 and one of dimension 9.

Next, in this case, $N_G(P) = P \times Z(G) \cong P \times C_3$, hence by the definition of $K(G)$ we need to show that the Green correspondents X_1 and X_2 of the two non-trivial 1-dimensional $kN_G(P)$ -modules are endo-trivial kG -modules. However, as X_1 and X_2 must be dual to each other, it suffices to consider X_1 . Computing X_1 with MAGMA we find that X_1 has the given Loewy (and socle) series and that its restriction to P is an endo-trivial module and therefore so is X_1 .

- (c) For $Q \cong \mathrm{PGL}_2^*(9)$ we take $\tilde{Q} = 3.\mathrm{PGL}_2^*(9)$ and the claim follows from Theorem 3.6(b) together with (b). □

We can now prove our main Theorem.

Proof of Theorem 1.1. We know from Theorem 3.1(a)–(d) that

$$T(G) = TT(G) \oplus TF(G) \cong K(G) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z},$$

so that it only remains to compute the group $K(G)$ of trivial source endo-trivial modules. If $G/O_{2'}(G) \not\cong \mathrm{PGL}_2^*(9)$, then $K(G) = X(G)$ by Proposition 6.1, hence (a). If $G/O_{2'}(G) \cong \mathrm{PGL}_2^*(9)$, then the assertions in (b) follow directly from Lemma 6.2(a),(b) and (c). □

ACKNOWLEDGEMENTS

The authors would like to thank Jesper Grodal, Burkhard Külshammer, Gunter Malle, Nadia Mazza, Benjamin Sambale, Mandi Schaeffer Fry, Gernot Stroth, Jacques Thévenaz, and Rebecca Waldecker for helpful hints and discussions related to the content of this article. They are grateful to the referee for detailed comments and suggestions.

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