

# SIMPLE MODULES IN THE AUSLANDER-REITEN QUIVER OF PRINCIPAL BLOCKS WITH ABELIAN DEFECT GROUPS

SHIGEO KOSHITANI AND CAROLINE LASSUEUR

ABSTRACT. Given an odd prime  $p$ , we investigate the position of simple modules in the stable Auslander-Reiten quiver of the principal block of a finite group with non-cyclic abelian Sylow  $p$ -subgroups. In particular, we prove a reduction to finite simple groups. In the case that the characteristic is 3, we prove that simple modules in the principal block all lie at the end of their components.

## 1. INTRODUCTION

The position of simple modules in the stable Auslander-Reiten quiver of the group algebra  $kG$  over a field  $k$  of characteristic  $p$  of a finite group  $G$  of order divisible by  $p$  is a question that was partially investigated in the 1980's and the 1990's in a series of articles by different authors. We refer the reader in particular to [Kaw97, KMU00, KMU01, BU01] and the references therein. The aim of this note is to come back to the following question:

**Question A.** *Let  $B$  be a wild  $p$ -block of  $kG$ . Under which conditions do all simple  $B$ -modules lie at the end of their connected components in the stable Auslander-Reiten quiver of  $kG$ ?*

A main reason of interest in this question lies in the fact that a simple  $kG$ -module lies at the end of its component if and only if the heart of its projective cover is indecomposable.

In this article, we focus attention on the case in which the principal block  $B_0(kG)$  is of wild representation type with abelian defect groups and the prime  $p$  is odd. We recall that a  $p$ -block is of wild representation type if and only if its defect groups are neither cyclic, nor dihedral, nor semi-dihedral, nor generalised quaternion (see [Hum06, §8.9 Theorem]). Thus, when  $p$  is odd, this amounts to requiring that the  $p$ -rank of  $G$  is at least 2. Question A in the case that  $p = 2$  was treated by Kawata-Michler-Uno in [KMU00, Theorem 5]. We aim at extending their results and part of their methods to arbitrary primes. Further, we note that the cases when  $B_0(kG)$  is of finite or tame representation type are well-understood. In the former case, the distance of a simple module to the rim of its connected component (which is a tube of shape  $(\mathbb{Z}/e\mathbb{Z})A_m$ ) is a function of its position in

---

*Date:* November 23, 2017.

*2010 Mathematics Subject Classification.* 20C20, 16G70.

*Key words and phrases.* Stable Auslander-Reiten quiver, simple module, principal block, abelian defect.

The authors gratefully acknowledge financial support by the funding scheme TU Nachwuchsring Individual Funding 2016 granted to the second author by the TU Kaiserslautern. The first author was also supported by the Japan Society for Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C)15K04776, 2015–2018.

the Brauer tree of the block, while in the latter case the position of the simple modules in their connected components is given by Erdmann's work on tame blocks [Erd90].

Assuming the field  $k$  is algebraically closed we prove the following results:

**Theorem B.** *Let  $G$  be a finite group and  $N \trianglelefteq G$  a normal subgroup such that  $G/N$  is solvable of  $p'$ -order. Let  $B$  and  $b$  be wild blocks of  $kG$  and  $kN$  respectively such that  $1_B = 1_b$ . If every simple  $b$ -module lies at the end of its connected component in the stable Auslander-Reiten quiver of  $kN$ , then every simple  $B$ -module lies at the end of its connected component in the stable Auslander-Reiten quiver of  $kG$ .*

**Theorem C.** *Let  $p$  be an odd prime. Let  $G$  be a finite group with non-cyclic abelian Sylow  $p$ -subgroups and  $O_{p'}(G) = 1$ . Write  $O_{p'}(G) = Q \times H_1 \times \cdots \times H_m$  ( $m \geq 0$ ), where  $Q$  is an abelian  $p$ -group and  $H_i$  is a non-abelian finite simple group with non-trivial Sylow  $p$ -subgroups for each  $1 \leq i \leq m$ . Assume that one of the following conditions is satisfied:*

- (i)  $Q \neq 1$ ; or
- (ii)  $Q = 1$  and  $m \geq 2$ ; or
- (iii)  $Q = 1$ ,  $m = 1$  and every simple  $B_0(kH_1)$ -module lies at the end of its connected component in the stable Auslander-Reiten quiver of  $kH_1$ .

*Then every simple  $B_0(kG)$ -module lies at the end of its connected component in the stable Auslander-Reiten quiver of  $kG$ .*

**Corollary D.** *Let  $p$  be an odd prime. Assume that every simple  $B_0(kH)$ -module lies at the end of its connected component in the stable Auslander-Reiten quiver of  $kH$  for every non-abelian finite simple group  $H$  with non-cyclic abelian Sylow  $p$ -subgroups. Then every simple  $B_0(kG)$ -module lies at the end of its connected component in the stable Auslander-Reiten quiver of  $kG$  for any finite group  $G$  with non-cyclic abelian Sylow  $p$ -subgroups.*

We note that if  $p = 2$ , then the analogues of Theorem C and Corollary D were essentially proven by Kawata-Michler-Ueno [KMU00], although not stated in these terms. As a corollary, we also obtain the equivalent of [KMU00, Theorem 5(a)] for the prime 3.

**Theorem E.** *Assume  $p = 3$ . Let  $G$  be a finite group with abelian Sylow 3-subgroups. If  $B_0(kG)$  is a wild 3-block, then every simple  $B_0(kG)$ -module lies at the end of its connected component in the stable Auslander-Reiten quiver of  $kG$ .*

The paper is organised as follows: in Section 2, we set up the notation and in Section 3 we recall the state of knowledge on the subject and extend a result of Kawata's [Kaw97, Theorem 1.5] to describe more precisely the indecomposable summands of the heart of the projective cover of a simple module not lying on the rim of its component. In Section 4, we consider groups having a solvable quotient of  $p'$ -order and prove Theorem B. In Sections 5 and 6, we proceed to a reduction of Question A for principal blocks to the case of finite non-abelian simple groups and prove Theorem C and Corollary D. Finally in Section 7 we deal with the case  $p = 3$  and prove Theorem E.

## 2. NOTATION AND PRELIMINARIES ON MODULE AND BLOCK THEORY

Throughout this paper, unless otherwise stated, we adopt the following notation and conventions. We assume that  $k$  is an algebraically closed field of characteristic  $p > 0$ . All

groups are assumed to be finite and we let  $G$  denote a finite group of order divisible  $p$ . We let  $(\mathcal{K}, \mathcal{O}, k)$  be a splitting  $p$ -modular system for  $G$  and its subgroups, namely  $\mathcal{O}$  is a complete discrete valuation ring,  $\mathcal{K}$  its quotient field of characteristic 0,  $k = \mathcal{O}/J(\mathcal{O})$  its residue field of characteristic  $p$  (where  $J(\mathcal{O})$  is the unique maximal ideal of  $\mathcal{O}$ ), and  $\mathcal{K}$  and  $k$  are both splitting fields for all subgroups of  $G$ .

For a  $p$ -block  $B$ , we write  $1_B$  for the corresponding block idempotent and  $\text{IBr}(B)$  for the set of isomorphism classes of simple  $kB$ -modules. Furthermore, unless otherwise specified, we assume that  $B_0 = B_0(kG)$ , the principal block of  $kG$ , is wild. Thus, when the defect groups of  $B$  are abelian, we may therefore assume that a Sylow  $p$ -subgroup of  $G$  is non-cyclic, or equivalently that the  $p$ -rank of  $G$  is at least 2.

We write  $H \leq G$  if  $H$  is a subgroup of  $G$ ,  $N \trianglelefteq G$  if  $N$  is a normal subgroup of  $G$ ,  $O_{p'}(G)$  for the largest  $p'$ -normal subgroup of  $G$ , and  $O^{p'}(G)$  for the smallest normal subgroup of  $G$  with  $p'$ -index in  $G$ . All modules are assumed to be finitely generated right modules. We denote by  $k_G$  the trivial  $kG$ -module. If  $H \leq G$  and  $X, Y$  are  $kG$ - and  $kH$ -modules respectively, then we write  $X \downarrow_H$  for the restriction of  $X$  to  $H$ , and  $Y \uparrow^G = Y \otimes_{kH} kG$  for the induction of  $Y$  from  $H$  to  $G$ .

We let  $J = J(kG)$  denote the Jacobson radical of  $kG$ . For a  $kG$ -module  $U$ , we define  $J(kG)^0 = kG$  and for any non-negative integer  $i \geq 0$  we let  $\text{soc}^i(U) = \{u \in U \mid uJ^i = \{0\}\}$ , then inductively for any  $i \geq 1$ , we write

$$L_i(U) = UJ^{i-1}/UJ^i \quad \text{and} \quad S_i(U) = \text{soc}^i(U)/\text{soc}^{i-1}(U)$$

for the  $i$ -th *Loewy layer* and the  $i$ -th *socle layer* of  $U$ , respectively. We define  $\text{soc}(U) = \text{soc}^0(U)$ , that is called the socle of  $U$  (see [Lan83, Chapter I Definition 8.1]). For an integer  $n \geq 1$  and simple  $kG$ -modules  $S_1, \dots, S_n$  (possibly  $S_i \cong S_j$  for  $i \neq j$ ) we denote by

$$U = \begin{array}{|c|} \hline S_1 \\ S_2 \\ \vdots \\ S_n \\ \hline \end{array}$$

a uniserial  $kG$ -module of Loewy length  $n$  such that  $L_i(U) \cong S_i$  for every  $1 \leq i \leq n$ . We also recall that the Loewy series does not determine modules up to isomorphism. For instance if  $p = 2$  and  $G = C_2 \times C_2$ , then  $kG \cong k[x, y]/(x^2, y^2)$  as  $k$ -algebras and there are obviously two *non-isomorphic* uniserial  $kG$ -modules  $U$  and  $V$  of Loewy length 2 ( $\dim_k[\text{Ext}_{kG}^1(k, k)] = 2$ ). We will use throughout the following well-known properties without further mention:

**Lemma 2.1.** *Assume  $N \trianglelefteq G$  of index prime to  $p$ .*

- (a) *We have  $J = \tilde{J}kG = kG\tilde{J}$  where  $\tilde{J} = J(kN)$ .*
- (b) *Let  $X$  be a  $kG$ -module and  $Y$  a  $kN$ -module, then for any  $i \geq 1$  we have*

$$L_i(X) \downarrow_N = L_i(X \downarrow_N) \quad \text{and} \quad S_i(X) \downarrow_N = S_i(X \downarrow_N)$$

and

$$L_i(Y) \uparrow^G = L_i(Y \uparrow^G) \quad \text{and} \quad S_i(Y) \uparrow^G = S_i(Y \uparrow^G).$$

*Proof.* Part (a) is a well-known result of Villamayor [Vil59] and part (b) follows from (a).  $\square$

Given a normal subgroup  $H \trianglelefteq G$  and  $\tilde{b}$  a  $p$ -block of  $H$ , we will use the group  $G[\tilde{b}]$  defined by Dade [Dad73]. A more explicit description of  $G[\tilde{b}]$  can also be found in [HK99, Lemma 3.2]. Roughly speaking  $G[\tilde{b}]$  is the stabiliser of  $\tilde{b}$  as  $k(H \times H)$ -module.

**Lemma 2.2.** (Dade) *Let  $H \trianglelefteq G$  such that  $p \nmid |G/H|$ , and let  $P$  be a Sylow  $p$ -subgroup of  $H$ . Let  $\tilde{b} = B_0(kH)$  and  $B = B_0(kG)$  be the principal blocks of  $kH$  and  $kG$ , respectively. Set  $N = HC_G(P)$ . Then the following hold:*

- (a) *The block  $\tilde{b}$  is  $G$ -invariant.*
- (b)  *$N = G[\tilde{b}]$  and  $N \trianglelefteq G$ .*
- (c) *If  $b$  denotes the principal block of  $kN$ , then  $1_B = 1_b$ .*

*Proof.* (a) Obvious since  $\tilde{b}$  is the principal block.

(b) The first claim follows from [Dad73, Corollary 12.6] since  $\tilde{b}$  is the principal block. Moreover, as  $\tilde{b}$  is  $G$ -invariant, the fact that  $G[\tilde{b}] \trianglelefteq G$  follows from [Dad73, Proposition 2.17].

(c) The main argument to prove (c) is given by [Kue95, p.303 line 10]). We give here a full argument for completeness. As  $\tilde{b}$  is  $G$ -invariant,  $1_{\tilde{b}}$  is an idempotent of  $Z(kG)$  and we can write

$$1_{\tilde{b}} = 1_B + 1_{B_1} + \cdots + 1_{B_n}$$

for an integer  $n \geq 0$  and for distinct non-principal blocks  $B_1, \dots, B_n$  of  $kG$ . Thus,  $1_{\tilde{b}} 1_B = 1_B$ . Namely,

$$1_B \in 1_{\tilde{b}} Z(kG) \subseteq 1_{\tilde{b}} C_{kG}(H) =: C.$$

This implies  $1_B \in Z(C)$  since  $1_B \in Z(kG)$ . Hence it follows from [Kue90, Corollary 4] and part (b) that

$$\begin{aligned} 1_B \in C[\tilde{b}] &= Z(\tilde{b}) * G[\tilde{b}] \\ &\subseteq Z(kH) * N \subseteq kN, \end{aligned}$$

where  $*$  denotes the crossed product. Thus  $1_B \in Z(kN)$ . On the other hand, since  $b$  is the principal block of  $kN$ , we have  $1_b$  is  $G$ -invariant, so that  $1_b \in Z(kG)$ . Hence, as above, we can write

$$1_b = 1_B + 1_{B'_1} + \cdots + 1_{B'_t}$$

where  $t \geq 0$  is an integer and  $B'_1, \dots, B'_t$  are distinct non-principal blocks of  $kG$ . Set  $\tilde{e} = 1_b - 1_B \in Z(kN)$  (since  $1_B \in Z(kN)$ ). Therefore  $1_b = 1_B + \tilde{e}$  is a decomposition of  $1_b$  into orthogonal idempotents of  $Z(kN)$ . This implies that  $\tilde{e} = 0$ , and hence  $1_b = 1_B$ .  $\square$

Finally, we will need the following well-known properties of relative projectivity and inflation in direct products. We recall that if  $N \trianglelefteq G$  and  $U$  is a  $k(G/N)$ -module, then we denote by  $\text{Inf}_{G/N}^G(U)$  the inflation of  $U$  from  $G/N$  to  $G$ , namely  $\text{Inf}_{G/N}^G(U) = U$  as  $k$ -vector space and becomes a  $kG$ -module via the action of  $G$  obtained by composition with the canonical epimorphism  $G \twoheadrightarrow G/N$ . Furthermore, if  $G = N \times H$  is the direct product

of two finite groups  $N$  and  $H$ , and  $U$  and  $V$ , are  $kN$ - and  $kH$ -modules respectively, then on the one hand  $X = U \otimes_k V$  becomes a  $kG$ -module via the action

$$(u \otimes v)(n, h) = un \otimes vh \quad \forall u \in U, v \in V, n \in N, h \in H,$$

and on the other hand, setting  $U' = \text{Inf}_{G/H \cong N}^G(U)$  and  $V' = \text{Inf}_{G/N \cong H}^G(V)$ , we have that  $U' \otimes_k V'$  becomes a  $kG$ -module via the diagonal action

$$(u' \otimes v') \cdot g = u'g \otimes v'g \quad \forall u' \in U', v' \in V', g \in G.$$

It is then easily seen that  $X \cong U' \otimes_k V'$  as  $kG$ -modules.

**Lemma 2.3.** *Assume that  $G = N \times H$  is the direct product of two finite groups  $N$  and  $H$ . Let  $U$  be a  $kN$ -module and  $V$  be a  $kH$ -module. Set  $X = U \otimes_k V$ ,  $U' = \text{Inf}_{G/N}^G(U)$  and  $V' = \text{Inf}_{G/H}^G(V)$ . If  $U$  is projective as a  $kL$ -module, then  $X$  is a relatively  $H$ -projective  $kG$ -module.*

We give a short proof for completeness.

*Proof.* First, since  $U$  is a projective  $kN$ -module, it is projective relatively to the trivial subgroup, or equivalently projective relatively to the  $kN$ -module  $k_{\{1\}} \uparrow^N$ , that is projective relatively to the  $k(G/H)$ -module  $k_{H/H} \uparrow^{G/H}$  by using the isomorphism  $N \cong G/H$ . Therefore, by [Las11, Lemma 2.1.1(c)], it follows that the inflated  $kG$ -module  $U' = \text{Inf}_{G/H}^G(U)$  is projective relatively to the inflated  $kG$ -module

$$\begin{aligned} \text{Inf}_{G/H}^G \left( k_{H/H} \uparrow^{G/H} \right) &= \text{Inf}_{G/H}^G \circ \text{Ind}_{H/H}^{G/H} (k_{G/H}) \\ &= \text{Ind}_H^G \circ \text{Inf}_{H/H}^H (k_{H/H}) = k_H \uparrow^G \end{aligned}$$

(where  $\text{Ind}$  denotes the induction seen as a functor). Hence  $U'$  is projective relatively to  $H$ . It follows directly that the tensor product  $X \cong U' \otimes_k V'$  is  $H$ -projective, because one of the factors is (see e.g. [Ben98, Corollary 3.6.7]).  $\square$

### 3. BACKGROUND RESULTS ON THE AUSLANDER-REITEN QUIVER

We recall briefly basic facts concerning the stable Auslander-Reiten quiver of a group algebra, which we will be using in the sequel. For a complete introduction to Auslander-Reiten theory, we refer the reader to the textbooks [ARS95, Chapter IV] and [Ben98, Chapter 4].

To finish setting up our notation, given a  $kG$ -module  $M$ , we denote by  $\Omega^n(M)$  ( $n \in \mathbb{Z}$ ) its  $n$ -th Heller translate of  $M$ . Given a simple  $kG$ -module  $S$ , we denote by  $P(S)$  its projective cover and by  $\mathcal{H}(P(S))$  the heart of  $P(S)$ , that is  $\mathcal{H}(P(S)) = P(S)J / \text{soc}(P(S))$ .

Let  $M$  be an indecomposable  $kG$ -module. By definition, an *Auslander-Reiten sequence* (or *AR-sequence*) terminating at  $M$  is a non-split short exact sequence

$$\mathcal{A}(M) : \quad 0 \longrightarrow N \xrightarrow{f} X_M \xrightarrow{g} M \longrightarrow 0$$

satisfying the following conditions: firstly  $N$  is indecomposable, and secondly for each  $kG$ -homomorphism  $h : X \rightarrow M$  which is not a split epimorphism, there exists a  $kG$ -homomorphism  $h' : X \rightarrow E$  such that  $h = gh'$ . Given an indecomposable non-projective  $kG$ -module  $M$ , there exists always an *AR-sequence* terminating at  $M$ , and it is unique

up to isomorphism of short exact sequences. Moreover, since  $kG$  is a finite-dimensional symmetric Algebra, we have  $N \cong \Omega^2(M)$  (see [Ben98, 4.12.8]). In similar fashion, there exists an AR-sequence starting at  $M$ , unique up to isomorphism of short exact sequences, with end term isomorphic to  $\Omega^{-2}(M)$ . For a non-projective simple  $kG$ -module  $S$ , the Auslander-Reiten sequence terminating at  $\Omega^{-1}(S)$  is of the form

$$\mathcal{A}(\Omega^{-1}(S)) : 0 \rightarrow \Omega(S) \rightarrow \mathcal{H}(P(S)) \oplus P(S) \rightarrow \Omega^{-1}(S) \rightarrow 0$$

and is called the *standard sequence associated to  $S$* . This is the unique AR-sequence in which the PIM  $P(S)$  occurs.

The *Auslander-Reiten quiver* (or *AR-quiver*) of  $kG$  (resp. of a  $p$ -block  $B$  of  $kG$ ) is the directed graph  $\Gamma(kG)$  (resp.  $\Gamma(B)$ ) whose vertices are the isomorphism classes of indecomposable  $kG$ -modules (resp.  $B$ -modules), and the number of arrows between two indecomposable modules  $M$  and  $N$  corresponds to the dimension of the space of *irreducible maps* between  $M$  and  $N$ . We refer the reader to [ARS95, Chapter IV] for a precise definition. Then the *stable Auslander-Reiten quiver* of  $kG$  (resp. of  $B$ ) is obtained from  $\Gamma(kG)$  (resp.  $\Gamma(B)$ ) by removing the vertices corresponding to projective modules and all arrows attached to these vertices; it is denoted by  $\Gamma_s(kG)$  (resp.  $\Gamma_s(B)$ ). By convention, we use the terminology *AR-component* to refer to a connected component of  $\Gamma_s(kG)$ , and we denote by  $\Gamma_s(M)$  the connected component of  $\Gamma_s(kG)$  containing a given indecomposable  $kG$ -module  $M$ .

Erdmann [Erd95] proved that all components of the stable Auslander-Reiten quiver belonging to a wild block have tree class  $A_\infty$ , that is are of the form  $\mathbb{Z}A_\infty$  or infinite tubes  $\mathbb{Z}A_\infty/\langle \tau^a \rangle$  of rank  $a$ , where  $\tau = \Omega^2$  is the Auslander-Reiten shift. In a component with tree class  $A_\infty$  an indecomposable non-projective  $kG$ -module  $M$  is said to lie *at the end* (or *on the rim*) of its AR-component if the projective-free part of the middle term  $X_M$  of the Auslander-Reiten sequence

$$\mathcal{A}(M) : 0 \rightarrow \Omega^2(M) \rightarrow X_M \rightarrow M \rightarrow 0$$

terminating at  $M$  is indecomposable. In this set up, clearly a simple module  $S$  lies at the end of its component if and only if  $\mathcal{H}(P(S))$  is indecomposable, and  $S$  lies in a tube if and only if  $S$  is periodic (i.e.  $\Omega$ -periodic). We also recall that for a selfinjective algebra the shape of the components of the stable Auslander-Reiten quiver is an invariant of its Morita equivalence class. By the above, the property of lying on the rim of its AR-component for a non-projective simple module is also invariant under Morita equivalence.

Simple  $kG$ -modules are known to lie on the rim of their AR-components in the following cases:

**Theorem 3.1.** *Let  $B$  be a wild  $p$ -block of  $kG$ . Then every simple  $B$ -module lies at the end of its AR-component in all of the following cases:*

- (a)  $G$  has a non-trivial normal  $p$ -subgroup ([Kaw97, Theorem 2.1]);
- (b)  $G$  is  $p$ -solvable ([Kaw97, Corollary 2.2]);
- (c)  $G$  is a perfect finite group of Lie type in the defining characteristic and  $B$  has full defect ([KMU01, Theorem]);
- (d)  $G$  has an abelian Sylow 2-subgroup and  $B$  is the principal 2-block ([KMU00, Theorem 5]);

- (e)  $G$  is a symmetric group or an alternating group ([BU01, Theorems 5.3 and 5.5]);
- (f)  $p = 2$  and  $G$  is a Schur cover of a symmetric group or of an alternating group ([BU01, Theorems 5.3 and 5.5]);
- (g)  $p \neq 2$  and  $G$  is a Schur cover of a symmetric group or of an alternating group such that the defect of  $B$  is at least 3 ([BU01, Theorems 5.3 and 5.5]).

Moreover, we will use the following computational criterion throughout:

**Theorem 3.2** (Kawata's Criterion on Cartan matrices [Kaw97, Theorem 1.5]). *Let  $B$  be a wild  $p$ -block of  $kG$ . Suppose that there exists a simple  $B$ -module  $S$  lying on the  $n$ -th row from the end of  $\Gamma_s(S)$ , where  $n \geq 2$  is minimal with this property. Then there exist pairwise non-isomorphic simple  $B$ -modules  $S_2, \dots, S_n$  with the following properties:*

- (a) For each  $2 \leq i \leq n$ , we have that  $S_i \cong \Omega^{2(i-2)}(S_2)$  and  $S_i$  lies at the end of  $\Gamma_s(\Omega(S))$ ;
- (b) The projective covers of  $P(S_i)$  of the simple modules  $S_i$  ( $2 \leq i \leq n$ ) are uniserial of length  $n + 1$  with the following Loewy structure:

$$P(S_2) = \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline S \\ \hline S_2 \\ \hline \end{array}, \quad P(S_3) = \begin{array}{|c|} \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline S \\ \hline S_2 \\ \hline S_3 \\ \hline \end{array}, \quad \dots, \quad P(S_n) = \begin{array}{|c|} \hline S_n \\ \hline S \\ \hline S_2 \\ \hline \vdots \\ \hline S_{n-1} \\ \hline S_n \\ \hline \end{array}.$$

The Cartan matrix of  $B$  is given by

$$\begin{pmatrix} 2 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 2 & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 2 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & \cdots & 1 & * & \cdots & \cdots & * \\ 0 & \cdots & \cdots & 0 & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & * & \cdots & \cdots & * \end{pmatrix},$$

where the columns are labelled by  $S_n, \dots, S_2, S, \dots$  in this order.

*Remark 3.3.* (a) If the Cartan matrix of a block has the shape of Theorem 3.2(b) above with  $n = 2$ , then the simple module  $S$  corresponding to the second column lies on the 2nd row of its AR-component. Indeed in this case  $P(S_2) = \begin{array}{|c|} \hline S_2 \\ \hline S \\ \hline S_2 \\ \hline \end{array}$  and the standard sequence associated to  $S_2$  is

$$0 \rightarrow \Omega(S_2) \rightarrow S \oplus P(S_2) \rightarrow \Omega^{-1}(S_2) \rightarrow 0,$$

so that  $S_2$  lies at the end of its AR-component and  $S$  on the 2nd row of its AR-component.

A converse to Kawata's Criterion need not be true in general for an  $n \geq 3$ .

- (b) The above was used to produce two counter-examples of simple modules not lying at the end of their AR-components. Namely, the group  $F_4(2)$  for  $p = 5$  has a simple module in the principal block of dimension 875823 lying on the 2nd row of its AR-component, and the group  $2.Ru$  for  $p = 3$  has a faithful simple module also lying on the 2nd row. See [KMU01, §4]. Both counter-examples are obtained thanks to the decomposition matrices of these groups computed by G. Hiß.

We can now improve Kawata's result by describing more accurately the structure of the heart of the projective cover of the simple module  $S$  lying on the  $n$ -th row of its AR-component.

**Corollary 3.4.** *With the assumptions and the notation of Theorem 3.2, we have that the heart of the projective cover of the simple module  $S$  is decomposable and has a uniserial indecomposable summand of length  $n - 1$ . More precisely*

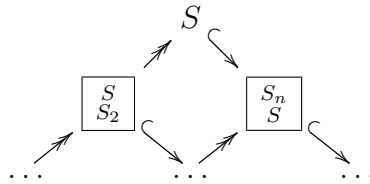
$$\mathcal{H}(P(S)) = \begin{bmatrix} S_2 \\ S_3 \\ \vdots \\ S_n \end{bmatrix} \oplus V,$$

where  $V$  is an indecomposable  $kG$ -module.

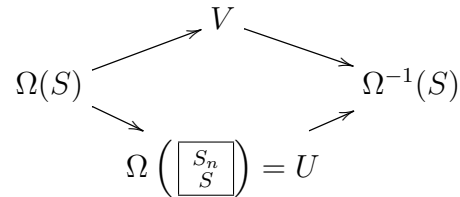
*Proof.* First, since by assumption the module  $S$  does not lie at the end of its AR-component, which is of tree class  $A_\infty$ , it is clear that the heart of the projective cover of  $S$  must be the direct sum of two indecomposable direct summands, say

$$\mathcal{H}(P(S)) \cong U \oplus V.$$

Using [Kaw97, Proposition 1.4], we have that the part of the component  $\Gamma_s(S)$  below  $S$  is as follows:



Therefore, using the fact that the Heller operator  $\Omega$  induces a graph isomorphism from  $\Gamma_s(S)$  to  $\Gamma_s(\Omega(S))$ , we have that in  $\Gamma_s(\Omega(S))$  the diamond corresponding to the standard sequence associated to  $S$  is as follows:



Hence it suffices to compute  $\Omega \left( \begin{bmatrix} S_n \\ S \end{bmatrix} \right)$ .



Since the top of the uniserial module  $\begin{bmatrix} S_n \\ S \end{bmatrix}$  is  $S_n$ , the projective cover of  $\begin{bmatrix} S_n \\ S \end{bmatrix}$  must be  $P(S_n)$ . Hence taking the kernel of the canonical surjection  $P(S_n) \twoheadrightarrow \begin{bmatrix} S_n \\ S \end{bmatrix}$ , we obtain from the Loewy series of the PIM  $P(S_n)$  in Theorem 3.2(c) that  $U$  has the following Loewy series

$$U = \Omega \left( \begin{bmatrix} S_n \\ S \end{bmatrix} \right) = \begin{bmatrix} S_2 \\ S_3 \\ \vdots \\ S_n \end{bmatrix}.$$

□

#### 4. GROUPS HAVING A SOLVABLE QUOTIENT OF $p'$ -ORDER

Throughout this section, we will assume that the following hypotheses hold:

**Hypothesis 4.1.** Assume that:

- (a)  $G$  is a finite group of order divisible by  $p$  and  $N \trianglelefteq G$  is a normal subgroup such that  $|G/N| =: q$  is a prime number with  $q \neq p$ , and we set  $G/N =: \langle gN \rangle$  for an element  $g \in G \setminus N$ .
- (b)  $B$  and  $b$  are wild blocks of  $kG$  and  $kN$  respectively such that  $1_B = 1_b$ .

**Lemma 4.2.** *Assume Hypothesis 4.1 holds. Let  $\zeta \in k^\times$  be a primitive  $q$ -th root of unity in  $k$ , and for each  $1 \leq j \leq q$  let  $Z_j$  be the one-dimensional  $k(G/N)$ -module defined by  $Z_j = \langle \alpha_j \rangle_k$  and  $\alpha_j \cdot gN = \zeta^{j-1} \alpha_j$ , so that in particular  $Z_1 = k_{G/N}$ . The following holds:*

- (a) *If  $\mathcal{S} \in \text{IBr}(B)$  is such that  $\mathcal{S} \downarrow_N$  is not simple, then for each  $1 \leq j \leq q$ ,*

$$\mathcal{S} \otimes_k Z_j \cong \mathcal{S}$$

*as  $kG$ -modules, where we see  $Z_j$  as a  $kG$ -module via inflation.*

- (b) *There are integers  $m \geq 1$  and  $\ell \geq 0$  such that*

$$\text{IBr}(B) = \{\mathcal{S}_{ij} \mid 1 \leq i \leq m; 1 \leq j \leq q\} \bigsqcup \{\mathcal{S}_i \mid m+1 \leq i \leq m+\ell\} \text{ and}$$

$$\text{IBr}(b) = \{\mathcal{T}_i \mid 1 \leq i \leq m\} \bigsqcup \{\mathcal{T}_{ij} \mid m+1 \leq i \leq m+\ell, 1 \leq j \leq q\},$$

*where for each  $1 \leq i \leq m$  and each  $1 \leq j \leq q$ ,*

$$\mathcal{S}_{ij} \downarrow_N = \mathcal{T}_i \quad \text{and} \quad \mathcal{T}_i \uparrow^G = \mathcal{S}_{i1} \oplus \cdots \oplus \mathcal{S}_{iq},$$

*and for each  $m+1 \leq i \leq m+\ell$  and each  $1 \leq j \leq q$ ,*

$$\mathcal{S}_i \downarrow_N = \mathcal{T}_{i1} \oplus \mathcal{T}_{i2} \oplus \cdots \oplus \mathcal{T}_{iq} \quad \text{and} \quad \mathcal{T}_{ij} \uparrow^G = \mathcal{S}_i$$

*where we may assume that  $\mathcal{T}_{ij} = \mathcal{T}_{i1}^{g^{j-1}}$ .*

*Moreover, we can assume that for each  $1 \leq j \leq q$ ,*

$$\mathcal{S}_{ij} = \mathcal{S}_{i1} \otimes_k Z_j.$$

*Proof.* (a) Let  $1 \leq j \leq q$ . By assumption and Clifford's theory we have that

$$\begin{aligned} (\mathcal{S} \otimes_k Z_j) \downarrow_N &= \mathcal{S} \downarrow_N \otimes_k k_N \cong \mathcal{S} \downarrow_N. \\ &= \mathcal{T} \oplus \mathcal{T}^g \oplus \cdots \oplus \mathcal{T}^{g^{q-1}} \end{aligned}$$

for some  $\mathcal{T} \in \text{IBr}(b)$ . Hence  $\mathcal{T} \uparrow^G \cong \mathcal{S}$ , and  $\mathcal{T} \uparrow^G \cong \mathcal{S} \otimes_k Z_j$  for each  $1 \leq j \leq q$ .

(b) As by Hypothesis 4.1 the quotient  $G/N$  is cyclic, the claim follows from the result of Schur-Clifford [NT88, Chapter 3 Corollary 5.9 and Problem 11(i)].  $\square$

**Lemma 4.3.** *Assume Hypothesis 4.1 holds. Let  $\mathcal{S} \in \text{IBr}(B)$ .*

- (a) *If  $\mathcal{S} \downarrow_N =: \mathcal{T}$  is simple, then  $P(\mathcal{S}) \downarrow_N \cong P(\mathcal{T})$  and  $\mathcal{H}(P(\mathcal{S})) \downarrow_N \cong \mathcal{H}(P(\mathcal{T}))$ .*  
 (b) *If  $\mathcal{S} \downarrow_N$  is not simple, then we can write  $\mathcal{S} \downarrow_N = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \cdots \oplus \mathcal{T}_q$  with  $\mathcal{T}_j = \mathcal{T}_1^{g^{j-1}}$  for each  $1 \leq j \leq q$  and we have that*

$$P(\mathcal{S}) \downarrow_N \cong P(\mathcal{T}_1) \oplus \cdots \oplus P(\mathcal{T}_q) \quad \text{and} \quad \mathcal{H}(P(\mathcal{S})) \downarrow_N \cong \bigoplus_{j=1}^q \mathcal{H}(P(\mathcal{T}_j)).$$

*Proof.* (a) Obviously

$$\begin{aligned} \mathcal{T} &= \mathcal{S} \downarrow_N = (P(\mathcal{S})/P(\mathcal{S})J) \downarrow_N = P(\mathcal{S}) \downarrow_N / (P(\mathcal{S})J) \downarrow_N \\ &= P(\mathcal{S}) \downarrow_N / (P(\mathcal{S})kG\tilde{J}) \quad \text{by Lemma 2.1} \\ &= P(\mathcal{S}) \downarrow_N / P(\mathcal{S}) \downarrow_N \tilde{J}. \end{aligned}$$

Hence the top of  $P(\mathcal{S}) \downarrow_N$  is  $\mathcal{T}$ , which implies that  $P(\mathcal{S}) \downarrow_N \cong P(\mathcal{T})$ . Therefore,

$$\mathcal{H}(P(\mathcal{S})) \downarrow_N = (P(\mathcal{S})J/\mathcal{S}) \downarrow_N = \mathcal{H}(P(\mathcal{T})).$$

(b) Similar to (a).  $\square$

**Proposition 4.4.** *Assume Hypothesis 4.1 holds. If every simple module  $T \in \text{IBr}(b)$  lies at the end of its AR-component, then every simple module  $S \in \text{IBr}(B)$  lies at the end of its AR-component.*

*Proof.* Let  $S \in \text{IBr}(B)$  be a simple module. First assume that  $S \downarrow_N =: T \in \text{IBr}(b)$  is simple. Then by Lemma 4.3(a)

$$\mathcal{H}(P(S)) \downarrow_N \cong \mathcal{H}(P(T)).$$

But by assumption  $\mathcal{H}(P(T))$  is indecomposable, therefore so is  $\mathcal{H}(P(S))$ .

We assume now for the rest of the proof that  $S \downarrow_N$  is not simple. If  $S$  lies at the end of its AR-component, then there is nothing to do. Therefore we now also assume that  $S$  lies on the  $n$ -th row from the bottom of  $\Gamma_s(S)$  for an integer  $n \geq 2$ , minimal (as in Kawata's Criterion on Cartan matrices). By Lemma 4.2(b),

$$S \downarrow_N = T_{11} \oplus \cdots \oplus T_{1q} \quad \text{and} \quad T_{1j} \uparrow^G = S \quad \text{for each } 1 \leq j \leq q,$$

where  $T_{1j} = T_{11}^{g^{j-1}}$  for  $1 \leq j \leq q$  are non-isomorphic simple modules in  $\text{IBr}(b)$ . We also set  $T_1 = T_{11}$ .

Let  $S_2, \dots, S_n$  be the simple modules given by Theorem 3.2.

**CLAIM 1.** If the modules  $S_2 \downarrow_N, \dots, S_n \downarrow_N$  are all non-simple, then we have a contradiction.

*Proof of Claim 1.* By assumption and Lemma 4.2, we can write

$$S_i \downarrow_N = T_{i1} \oplus T_{i2} \oplus \cdots \oplus T_{iq}.$$

For each  $2 \leq i \leq n$  we define  $T_i \in \text{IBr}(b)$  by  $T_{ij} = T_i^{g^{j-1}}$ , where  $1 \leq j \leq q$ . We claim that

$$P(T_2) = \begin{array}{|c|} \hline T_2 \\ \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline T_1 \\ \hline T_2 \\ \hline \end{array}, \quad P(T_3) = \begin{array}{|c|} \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline T_1 \\ \hline T_2 \\ \hline T_3 \\ \hline \end{array}, \quad \dots, \quad P(T_n) = \begin{array}{|c|} \hline T_n \\ \hline T_1 \\ \hline T_2 \\ \hline \vdots \\ \hline T_{n-1} \\ \hline T_n \\ \hline \end{array}.$$

Indeed, we know by Theorem 3.2(b) and Lemma 4.3(b) that

$$P(T_2) \oplus P(T_2)^g \oplus \dots \oplus P(T_2)^{g^{q-1}} = P(S_2) \downarrow_N = \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline S \\ \hline S_2 \\ \hline \end{array} \downarrow_N = \begin{array}{|c|} \hline T_2 \quad T_2^g \quad \dots \quad T_2^{g^{q-1}} \\ \hline T_3 \quad T_3^g \quad \dots \quad T_3^{g^{q-1}} \\ \hline \dots \\ \hline T_n \quad T_n^g \quad \dots \quad T_n^{g^{q-1}} \\ \hline T_1 \quad T_1^g \quad \dots \quad T_1^{g^{q-1}} \\ \hline T_2 \quad T_2^g \quad \dots \quad T_2^{g^{q-1}} \\ \hline \end{array},$$

where the boxes mean the Loewy and socle series of the  $kN$ -modules. Since the left-hand side is a direct sum of exactly  $q$  indecomposable  $kN$ -modules that are  $\langle g \rangle$ -conjugate to each other, by interchanging the indices of  $T_3, \dots, T_n, T_1$  desired, we may assume that the PIM  $P(T_2)$  has the desired structure. Then automatically the structures of  $P(T_3), \dots, P(T_n)$  are as claimed.

Now, using a similar argument as above, we also obtain

$$P(T_1) \oplus P(T_1)^g \oplus \dots \oplus P(T_1)^{g^{q-1}} = P(S) \downarrow_N = \begin{array}{|c|} \hline T_1 \oplus T_1^g \oplus \dots \oplus T_1^{g^{q-1}} \\ \hline \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline \end{array} \downarrow_N \oplus V \downarrow_N \\ \hline T_1 \oplus T_1^g \oplus \dots \oplus T_1^{g^{q-1}} \\ \hline \end{array},$$

for a  $kG$ -module  $V$  where the last equality holds by Corollary 3.4. Hence we have

$$\begin{aligned}
& \mathcal{H}(P(T_1)) \oplus \mathcal{H}(P(T_1))^g \oplus \cdots \oplus \mathcal{H}(P(T_1))^{g^{q-1}} \\
&= \begin{array}{|c|} \hline T_2 \oplus T_2^g \oplus \cdots \oplus T_2^{g^{q-1}} \\ \hline T_3 \oplus T_3^g \oplus \cdots \oplus T_3^{g^{q-1}} \\ \hline \vdots \\ \hline T_n \oplus T_n^g \oplus \cdots \oplus T_n^{g^{q-1}} \\ \hline \end{array} \oplus V \downarrow_N \\
&= \left( \begin{array}{|c|} \hline T_2 \\ \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline \end{array} \oplus \begin{array}{|c|} \hline T_2^g \\ \hline T_3^g \\ \hline \vdots \\ \hline T_n^g \\ \hline \end{array} \oplus \cdots \oplus \begin{array}{|c|} \hline T_2^{g^{q-1}} \\ \hline T_3^{g^{q-1}} \\ \hline \vdots \\ \hline T_n^{g^{q-1}} \\ \hline \end{array} \right) \oplus V \downarrow_N
\end{aligned}$$

since  $P(T_2), \dots, P(T_n)$  are uniserial by the above.

But we are assuming that  $T_2, \dots, T_n$  lie at the end of their AR-components, so that  $\mathcal{H}(P(T_2)), \dots, \mathcal{H}(P(T_n))$  are indecomposable. Further  $\mathcal{H}(P(T_1))$  is also indecomposable since  $T_1$  lies at the end of its AR-connected component. Therefore the right-hand side term in the latter equation has exactly  $q$  indecomposable direct summands. This implies that  $V = \{0\}$ , hence a contradiction.

CLAIM 2. If the modules  $S_2 \downarrow_N, \dots, S_n \downarrow_N$  are all simple, then we have a contradiction.

*Proof of Claim 2.* Set  $T_i = S_i \downarrow_N$  for  $2 \leq i \leq n$ . We have

$$S \downarrow_N = T_1 \oplus T_1^g \oplus \cdots \oplus T_1^{g^{q-1}}.$$

By the assumption and Lemma 4.2, for each  $2 \leq i \leq n$  we can write  $T_i \uparrow^G = S_{i1} \oplus \cdots \oplus S_{iq}$  with  $S_{ij} = S_{i1} \otimes_k Z_j$  for  $1 \leq j \leq q$ . In particular  $S_{i1} = S_i$  for each  $2 \leq i \leq n$ . By Theorem 3.2(b)

$$P(S_2) = \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline S \\ \hline S_2 \\ \hline \end{array},$$

so Lemma 4.2(a) implies that

$$P(S_{2j}) = P(S_2) \otimes_k Z_j = \begin{array}{|c|} \hline S_{2j} \\ \hline S_{3j} \\ \hline \vdots \\ \hline S_{nj} \\ \hline S \\ \hline S_{2j} \\ \hline \end{array} \quad \text{for } 1 \leq j \leq q.$$

These yield that  $P(S)$  has  $q$  distinct uniserial submodules

$$W_1 = \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline S \\ \hline \end{array}, \quad W_j = \begin{array}{|c|} \hline S_{2j} \\ \hline S_{3j} \\ \hline \vdots \\ \hline S_{nj} \\ \hline S \\ \hline \end{array} \quad \text{of Loewy length } n \text{ for } j = 2, \dots, q.$$

Set  $W = W_1 + W_2 + \dots + W_q \subseteq P(S)$ . Then  $\text{soc}(W) = S$ , and the Loewy and socle structure of  $W$  is as follows:

$$W = \begin{array}{|c|c|c|c|} \hline \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline \end{array} & \oplus & \begin{array}{|c|} \hline S_{22} \\ \hline S_{32} \\ \hline \vdots \\ \hline S_{n2} \\ \hline \end{array} & \oplus \dots \oplus & \begin{array}{|c|} \hline S_{2q} \\ \hline S_{3q} \\ \hline \vdots \\ \hline S_{nq} \\ \hline \end{array} \\ \hline & & S & & \\ \hline \end{array}$$

with simple socle isomorphic to  $S$ . Therefore  $W/S$  has a proper uniserial submodule

$$U = \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline \end{array}.$$

Now by Corollary 3.4,  $U \mid \mathcal{H}(P(S))$ , so that by Lemma 4.2(a)

$$\begin{aligned} \begin{array}{|c|} \hline S_{2j} \\ \hline S_{3j} \\ \hline \vdots \\ \hline S_{nj} \\ \hline \end{array} &= \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline \end{array} \otimes_k Z_j \\ &= (U \otimes_k Z_j) \mid (\mathcal{H}(P(S)) \otimes_k Z_j) \cong \mathcal{H}(P(S \otimes_k Z_j)) \cong \mathcal{H}(P(S)) \end{aligned}$$

for each  $1 \leq j \leq q$ . Therefore  $q = 2$  since  $\mathcal{H}(P(S))$  has exactly two non-projective indecomposable direct summands by the assumption that  $S$  does not lie at the end of its AR-component. Notice that this already provides a contradiction in case the characteristic of  $k$  is 2, since we assume  $q \neq p$ . So we now assume that  $p \geq 3$ . Then, the Loewy and socle structures of PIMs  $P(S)$ ,  $P(S_i)$  and  $P(S_{i2})$  for  $2 \leq i \leq n$  are:

$$\begin{array}{|c|c|} \hline S & \\ \hline \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline \end{array} & \begin{array}{|c|} \hline S_{22} \\ \hline S_{32} \\ \hline \vdots \\ \hline S_{n2} \\ \hline \end{array} \\ \hline S & \\ \hline \end{array}, \quad \begin{array}{|c|} \hline S_2 \\ \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline S \\ \hline S_2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline S_{22} \\ \hline S_{32} \\ \hline \vdots \\ \hline S_{n2} \\ \hline S \\ \hline S_2 \\ \hline S_{22} \\ \hline \end{array}, \quad \begin{array}{|c|} \hline S_3 \\ \hline \vdots \\ \hline S_n \\ \hline S \\ \hline S_2 \\ \hline S_3 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline S_{32} \\ \hline \vdots \\ \hline S_{n2} \\ \hline S \\ \hline S_{22} \\ \hline S_{32} \\ \hline \end{array}, \quad \dots, \quad \begin{array}{|c|} \hline S_n \\ \hline S \\ \hline S_2 \\ \hline \vdots \\ \hline S_{n-1} \\ \hline S_n \\ \hline \end{array}, \quad \begin{array}{|c|} \hline S_{n2} \\ \hline S \\ \hline S_{22} \\ \hline \vdots \\ \hline S_{n-1,2} \\ \hline S_{n,2} \\ \hline \end{array}.$$

Now considering the restrictions  $S \downarrow_N$  and  $S_i \downarrow_N$  for  $2 \leq i \leq n$ , we obtain by Lemma 4.3 that the Loewy and socle structures of the PIMs  $P(T_1)$ ,  $P(T_1^g)$  and  $P(T_i)$  for each  $2 \leq i \leq n$  are

$$\begin{array}{|c|} \hline T_1 \\ \hline T_2 \\ \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline T_1 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline T_1^g \\ \hline T_2 \\ \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline T_1^g \\ \hline \end{array}, \quad \begin{array}{|c|} \hline T_2 \\ \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline T_1 \oplus T_1^g \\ \hline T_2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline T_1 \oplus T_1^g \\ \hline T_2 \\ \hline T_3 \\ \hline \end{array}, \quad \dots, \quad \begin{array}{|c|} \hline T_n \\ \hline T_1 \oplus T_1^g \\ \hline T_2 \\ \hline T_3 \\ \hline \vdots \\ \hline T_n \\ \hline \end{array}$$

since  $T_1 \not\cong T_1^g$ . Now, as the dimension of any PIM for  $kN$  is divisible by  $|N|_p =: p^a$  for an integer  $a \geq 1$ , and since  $\dim T_1 = \dim T_1^g$ , we have for each  $2 \leq i \leq n$

$$0 \equiv \dim P(T_i) - \dim P(T_1) = \dim T_i \pmod{p^a},$$

so that

$$\begin{aligned} 0 &\equiv \dim P(T_1) \\ &\equiv \dim P(T_1) - (\dim T_2 + \dim T_3 + \dots + \dim T_n) = 2 \cdot \dim T_1 \pmod{p^a}. \end{aligned}$$

This implies that

$$\dim T_1 \equiv 0 \pmod{p^a}$$

since  $p \neq 2$  (since  $q = 2$ ). Thus,  $\dim T_i \equiv 0 \pmod{p^a}$  for any  $1 \leq i \leq n$ . Now, looking at the composition factors of the PIMs  $P(T_1), P(T_1^g), P(T_2), \dots, P(T_n)$ , we know that  $\text{IBr}(b) = \{T_1, T_1^g, T_2, \dots, T_n\}$ , which implies that  $p^a \mid \dim \mathcal{T}$  for any  $\mathcal{T} \in \text{IBr}(b)$ . Now it follows from Brauer's result [NT88, Chapter 3 Theorem 6.25] that there is a simple  $\mathcal{T} \in \text{IBr}(b)$  such that  $\nu_p(\dim \mathcal{T}) = a - d(b)$  (where  $d(b)$  is the defect of  $b$ ). Hence we have a contradiction since  $b$  is a wild block, i.e. of positive defect.

CLAIM 3.

- (a) If there is an integer  $2 \leq m \leq n-1$  such that  $S_2 \downarrow_N, \dots, S_m \downarrow_N$  are not simple and  $S_{m+1} \downarrow_N$  is simple, then we have a contradiction.
- (b) If there is an integer  $2 \leq m \leq n-1$  such that  $S_2 \downarrow_N, \dots, S_m \downarrow_N$  are simple and  $S_{m+1} \downarrow_N$  is not simple, then we have a contradiction.

*Proof of Claim 3.* (a) Set  $T_{m+1} = S_{m+1} \downarrow_N$ . By Lemma 4.2 there exists a simple module  $T_m \in \text{IBr}(b)$  with  $S_m \downarrow_N = T_m \oplus T_m^g \oplus \dots \oplus T_m^{g^{q-1}}$ . Then, by Lemma 4.2,

$$T_{m+1} \uparrow^G = S_{m+1} \oplus S_{m+1,2} \oplus \dots \oplus S_{m+1,q}$$

where  $S_{m+1,j} = S_{m+1} \otimes_k Z_j$  for each  $1 \leq j \leq q$  and  $T_m \uparrow^G = S_m$ . By the structure of  $P(S)$ , we have that  $\text{Ext}_{kG}^1(S_m, S_{m+1}) \neq 0$ . Therefore by Eckmann-Shapiro's Lemma we have that  $\text{Ext}_{kN}^1(T_m, T_{m+1}) \neq 0$ . Thus there exists a  $kN$ -module with Loewy structure

$$\begin{array}{|c|} \hline T_m \\ \hline T_{m+1} \\ \hline \end{array}.$$

So it follows from Lemma 2.1 that

$$\begin{array}{|c|} \hline T_m \\ \hline T_{m+1} \\ \hline \end{array} \uparrow^G = \begin{array}{|c|} \hline S_m \\ \hline S_{m+1} \oplus S_{m+1,2} \oplus \cdots \oplus S_{m+1,q} \\ \hline \end{array}$$

where the right-hand side box is the Loewy and socle series. But  $P(S_m)$  is uniserial by Theorem 3.2(b), so applying again Lemma 2.1, we must have  $q = 1$ , which contradicts the assumption that  $q$  is a prime.

(b) follows in a similar fashion using a dual argument.

Altogether, Claims 1-3 prove that the simple modules  $S_2, \dots, S_n$  cannot exist, therefore  $S$  must lie at the end of its AR-component.  $\square$

As a consequence of the above discussion we obtain Theorem B of the Introduction.

*Proof of Theorem B.* Because  $G/N$  is solvable of order prime to  $p$ , it follows by induction on  $|G/N|$ , that we may assume that  $|G/N|$  is a prime distinct from  $p$ . Then Proposition 4.4 yields the result.  $\square$

## 5. THE PRINCIPAL BLOCK OF $O^{p'}(G)$

From now on, we assume that  $p \geq 3$  and  $G$  is a finite group with non-trivial abelian Sylow  $p$ -subgroups. Because we consider the principal block only, we assume that  $O_{p'}(G) = 1$  since  $B_0(kG) \cong B_0(k(G/O_{p'}(G)))$  as  $k$ -algebras.

The structure of  $O^{p'}(G)$  can be obtained using the classification of finite simple groups and a result of Fong-Harris [FH93, 5A–5C].

**Lemma 5.1** ([Due04, Theorem 1.7]). *Let  $p$  be an odd prime. Let  $G$  be a finite group with a non-trivial abelian Sylow  $p$ -subgroup. Then*

$$O^{p'}(G/O_{p'}(G)) \cong Q \times H_1 \times \cdots \times H_m,$$

where  $m$  is a non-negative integer (i.e. possibly  $O^{p'}(G/O_{p'}(G)) \cong Q$ ),  $Q$  is an abelian  $p$ -group, and for each  $1 \leq i \leq m$ ,  $H_i$  is a non-abelian simple group with non-trivial Sylow  $p$ -subgroups.

Therefore, we fix the notation  $O^{p'}(G) = Q \times H_1 \times \cdots \times H_m$ , where  $Q$  is an abelian  $p$ -group, and  $H_1, \dots, H_m$  are non-abelian simple groups with non-trivial Sylow  $p$ -subgroups as given by Lemma 5.1.

### 5.1. Simple modules in infinite tubes $\mathbb{Z}A_\infty/\langle\tau^a\rangle$ .

**Lemma 5.2** ([KMU00, Lemma 5.2] generalised version). *Let  $H = \tilde{H}_1 \times \cdots \times \tilde{H}_m$  ( $m \geq 1$ ) be a finite group such that  $p \mid |\tilde{H}_i|$  for each  $1 \leq i \leq m$ . If  $B_0(kH)$  is a wild block and contains a periodic simple module, then  $m = 1$ .*

*Proof.* Let  $S$  be a simple periodic  $B_0(kH)$ -module. Then we may write  $S = S_1 \otimes_k \cdots \otimes_k S_m$  where  $S_i$  is a simple  $B_0(k\tilde{H}_i)$ -module for each  $1 \leq i \leq m$ . Then, by iterating [KMU00, Lemma 2.2], there exists an index  $1 \leq i_0 \leq m$  such that  $S_{i_0}$  is periodic and  $S_j$  is a projective  $k\tilde{H}_j$ -module for each  $1 \leq j \neq i_0 \leq m$ . But  $B_0(k\tilde{H}_j)$  cannot contain a simple

projective module, since we assume that  $p \mid |\tilde{H}_i|$  for each  $1 \leq i \leq m$ . Hence this forces  $H = \tilde{H}_{i_0}$ , i.e.  $m = 1$ .  $\square$

As a consequence, the existence of simple periodic modules in the principal block lying in tubes drastically restricts the possible structure of  $O^{p'}(G)$ .

**Corollary 5.3.** *If  $B_0(kG)$  contains a periodic simple module, then  $O^{p'}(G) = H_1$  is a non-abelian finite simple group with non-cyclic abelian Sylow  $p$ -subgroups.*

*Proof.* By Lemma 5.2, either  $O^{p'}(G) = Q$  or  $O^{p'}(G) = H_1$ . But the former cannot happen. Indeed, the indecomposable direct summands of the restriction to  $O^{p'}(G)$  of a simple periodic  $kG$ -module are all simple periodic modules, however the unique simple  $kQ$ -module is the trivial module, which is not periodic since we assume that  $B_0(kG)$  is wild, and hence  $Q$  is non-cyclic. The leaves only the possibility  $O^{p'}(G) = H_1$ , and the  $p$ -rank of  $H_1$  must be at least 2 again because we assume that  $B_0(kG)$  is wild.  $\square$

This immediately leads to the following reduction to non-abelian simple groups:

**Corollary 5.4.** *Assume that every periodic simple  $B_0(kH)$ -module lies at the end of its AR-component for every non-abelian finite simple group  $H$  with non-cyclic abelian Sylow  $p$ -subgroups. Then every simple periodic  $B_0(kO^{p'}(G))$ -module lies at the end of its AR-component for any finite group  $G$  with  $O_{p'}(G) = 1$  and non-cyclic abelian Sylow  $p$ -subgroups.*

## 5.2. Simple modules in $\mathbb{Z}A_\infty$ -components.

**Lemma 5.5.** *Let  $H = \tilde{H}_1 \times \cdots \times \tilde{H}_m$  ( $m \geq 1$ ) be a finite group with abelian Sylow  $p$ -subgroups such that  $p \mid |\tilde{H}_i|$  for each  $1 \leq i \leq m$ . If  $B_0(kH)$  is a wild block containing a non-periodic simple module  $S$  not lying at the end of its AR-component, then  $m = 1$ .*

This lemma and its proof below generalise parts of the proof of [KMU00, Theorem 5(i)].

*Proof.* Assume that  $m \geq 2$ . Then by Theorem 3.2(b), there exists a simple  $B_0(kH)$ -module  $T$  lying at the end of  $\Gamma_s(\Omega(S))$ . By Knörr's Theorem [Knö79, 3.7 Corollary], we know that the vertices of the simple modules in  $B_0(kH)$  are the Sylow  $p$ -subgroups of  $H$ , because they are abelian. Now by assumption  $\Gamma_s(S) \cong \mathbb{Z}A_\infty$  by [Erd95], which implies that all the modules in  $\Gamma_s(S)$  and  $\Gamma_s(\Omega(S))$  have the Sylow  $p$ -subgroups as their vertices by [OU94, Theorem]. So all the modules in  $\Gamma_s(S)$  and  $\Gamma_s(\Omega(S))$  are not projective relatively to the subgroup  $N = \tilde{H}_1 \times \cdots \times \tilde{H}_{m-1}$  as it does not contain a Sylow  $p$ -subgroup of  $H$ . Thus, as  $p \neq 2$ , all the simple direct summands of  $S \downarrow_N$  belong to blocks of defect zero by [KMU00, Lemma 1.4]. But

$$B_0(kH) = B_0(kN) \otimes_k B_0(k\tilde{H}_m)$$

and hence there exist a simple  $B_0(kN)$ -module  $S_0$  and a simple  $B_0(k\tilde{H}_m)$ -module  $S_m$  such that

$$S = \text{Inf}_{N \times \tilde{H}_m / 1 \times \tilde{H}_m}^H(S_0) \otimes_k \text{Inf}_{N \times \tilde{H}_m / N \times 1}^H(S_m).$$

Because  $S \downarrow_N \cong (\dim_k S_m) S_0$ , the module  $S_0$  is a projective  $kN$ -module by the above. Hence by Lemma 2.3  $S$  is relatively  $\tilde{H}_m$ -projective. This contradicts the fact that the vertices of  $S$  are the Sylow  $p$ -subgroups of  $H$ . Thus we conclude that  $S$  must lie at the end of  $\Gamma_s(S)$ .  $\square$



**Proposition 5.6.** *Let  $G$  be a finite group with  $O_{p'}(G) = 1$  and non-cyclic abelian Sylow  $p$ -subgroups. Assume moreover that one of Conditions (i), (ii), or (iii) of Theorem C is satisfied. Then every non-periodic simple  $B_0(kO_{p'}(G))$ -module lies at the end of its AR-component.*

*Proof.* We have  $O_{p'}(G) = Q$  or  $O_{p'}(G) = Q \times H_1 \times \cdots \times H_m$ , where  $Q$  is an abelian  $p$ -group and  $H_i$  is a non-abelian finite simple group with non-trivial Sylow  $p$ -subgroups for each  $1 \leq i \leq m$ .

If (i) holds, that is  $Q \neq 1$ , then by Theorem 3.1(a), all simple  $B_0(kO_{p'}(G))$ -modules lie at the end of their AR-components. Therefore, we assume for the rest of the proof that  $Q = 1$ .

Next if (ii) holds, that is  $m \geq 2$ , the claim follows from Lemma 5.5.

Finally if (iii) holds, that is  $O_{p'}(G) = H_1$ , then  $H_1$  must have a non-cyclic Sylow  $p$ -subgroup, therefore all simple  $B_0(kO_{p'}(G))$ -modules lie at the end of their AR-components by assumption.  $\square$

## 6. REDUCTION TO $O_{p'}(G)$

We continue assuming that  $G$  is a finite group with non-cyclic abelian Sylow  $p$ -subgroups such that  $O_{p'}(G) = 1$ , unless otherwise stated. We now prove that an answer to Question A is detected by restriction to the normal subgroup  $O_{p'}(G)$  of  $G$ .

We set  $H = O_{p'}(G)$ , let  $P \in \text{Syl}_p(H)$  be a Sylow  $p$ -subgroup, and set  $N = HC_G(P)$ . Moreover we set  $B = B_0(kG)$ ,  $b = B_0(kN)$  and  $\tilde{b} = B_0(kH)$ . Then  $N$  is Dade's Group  $G[\tilde{b}]$  and  $N \trianglelefteq G$ , see Lemma 2.2.

First of all Question A has an affirmative answer for the group  $N$  if and only if it has an affirmative answer for the group  $H$ .

**Lemma 6.1.** *With the above notation, every simple  $b$ -module lies at the end of its AR-component if and only if every simple  $\tilde{b}$ -module lies at the end of its AR-component.*

*Proof.* By the Alperin-Dade Theorem [Dad77, Theorem] (see [Alp76]), the blocks  $b$  and  $\tilde{b}$  are isomorphic as  $k$ -algebras, hence Morita equivalent. But for a simple module, lying at the end of its AR-component is a property preserved by Morita equivalence.  $\square$

**Proposition 6.2.** *If every simple  $\tilde{b}$ -module lies at the end of its AR-component, then every simple  $B$ -module lies at the end of its AR-component.*

*Proof.* Let  $S$  be a simple  $B$ -module and let  $T$  be a simple direct summand of  $S \downarrow_H$ . Then  $T$  is periodic if and only if  $S$  is. Therefore  $\Gamma_s(S) \cong \mathbb{Z}A_\infty$  if and only if  $\Gamma_s(T) \cong \mathbb{Z}A_\infty$ , and  $\Gamma_s(S)$  is an infinite tube with tree class  $A_\infty$  if and only if  $\Gamma_s(T)$  is an infinite tube with tree class  $A_\infty$ .

In case  $\Gamma_s(S) \cong \mathbb{Z}A_\infty$ , then  $S$  lies at the end of  $\Gamma_s(S)$  if and only if  $T$  lies at the end of  $\Gamma_s(T)$  by [KMU00, Lemma 1.5].

In case  $\Gamma_s(S)$  is an infinite tube with tree class  $A_\infty$ , then by Corollary 5.3,  $H$  is a non-abelian finite simple group with non-cyclic abelian Sylow  $p$ -subgroups. Now, by Schreier's conjecture (now proven by the Classification of Finite Simple Groups, see [GLS2,

Definition 2.1] and [GLS3, Theorem 7.1.1]), we know that  $G/H$  is a solvable  $p'$ -subgroup of  $\text{Out}(H)$ . Now by Lemma 6.1, we may assume  $H = N$  and by Lemma 2.2(c) we have  $1_B = 1_b$ . Therefore Theorem B implies that  $S$  lies at the end of  $\Gamma_s(S)$  because every simple  $b$ -module lies at the end of its AR-component.  $\square$

As a corollary, we obtain Theorem C of the Introduction.

*Proof of Theorem C.* Let  $G$  be a finite group with non-cyclic abelian Sylow  $p$ -subgroups. As  $B_0(kG) \cong B_0(kG/O_{p'}(G))$  as  $k$ -algebras, we may assume that  $O_{p'}(G) = 1$ . Therefore, by Proposition 6.2, every simple  $B_0(kG)$ -module lies at the end of its AR-component if every simple  $B_0(kO_{p'}(G))$ -module lies at the end of its AR-component. Now if  $B_0(kG)$  contains a periodic simple module, then by Corollary 5.3 we must have that  $O_{p'}(G) = H_1$  is a non-abelian finite simple group with non-cyclic abelian Sylow  $p$ -subgroups, then the claim holds by Corollary 5.4. Therefore we may assume that  $B_0(kG)$ , and hence  $B_0(kO_{p'}(G))$ , contains no periodic simple module. In this case, if one of Conditions (i),(ii), or (iii) holds, then the claim follows from Proposition 5.6.  $\square$

Now Corollary D is a direct consequence of Theorem C.

## 7. PRINCIPAL 3-BLOCKS

We now fix  $p = 3$ , and continue assuming that  $G$  is a finite group with non-cyclic Sylow 3-subgroups, so that  $B_0(kG)$  is wild. We may also assume that  $O_{3'}(G) = 1$ .

We start by investigating principal 3-blocks of non-abelian finite simple groups with abelian defect group. To this aim, we recall that the list of non-abelian finite simple groups with abelian Sylow 3-subgroups is known by the classification of finite simple groups and was determined by Paul Fong (in an unpublished manuscript).

**Proposition 7.1** ([KY10, Proposition 4.3]). *If  $G$  is a non-abelian finite simple group with non-cyclic abelian Sylow 3-subgroup, then  $G$  is one of:*

- (i)  $\mathfrak{A}_7, \mathfrak{A}_8, M_{11}, M_{22}, M_{23}, HS, O'N$ ;
- (ii)  $\text{PSL}_3(q)$  for a prime power  $q$  such that  $3 \mid (q-1)$ ;
- (iii)  $\text{PSU}_3(q^2)$  for a prime power  $q$  such that  $3 \mid (q+1)$ ;
- (iv)  $\text{PSp}_4(q)$  for a prime power  $q$  such that  $3 \mid (q-1)$ ;
- (v)  $\text{PSp}_4(q)$  for a prime power  $q$  such that  $q > 2$  and  $3 \mid (q+1)$ ;
- (vi)  $\text{PSL}_4(q)$  for a prime power  $q$  such that  $q > 2$  and  $3 \mid (q+1)$ ;
- (vii)  $\text{PSU}_4(q^2)$  for a prime power  $q$  such that  $3 \mid (q-1)$ ;
- (viii)  $\text{PSL}_5(q)$  for a prime power  $q$  such that  $3 \mid (q+1)$ ;
- (ix)  $\text{PSU}_5(q^2)$  for a prime power  $q$  such that  $3 \mid (q-1)$ ; or
- (x)  $\text{PSL}_2(3^n)$  for an integer  $n \geq 2$ .

As a consequence we obtain:

**Proposition 7.2.** *If  $G$  is a non-abelian finite simple group with non-cyclic abelian Sylow 3-subgroups, then every simple  $B_0(kG)$ -modules lies at the end of its component in  $\Gamma_s(B_0(kG))$ .*

*Proof.* Let  $P \in \text{Syl}_3(G)$ , and set  $N = N_G(P)$  and  $B_0 = B_0(kG)$ . We go through the list of groups in Proposition 7.1.

In case (i), in all cases all simple  $B_0$ -modules lie at the end of their component in  $\Gamma_s(B_0)$  by Theorem 3.2(b): indeed if  $G$  is one of  $\mathfrak{A}_8$ ,  $M_{22}$  or  $O'N$ , then one checks from GAP [GAP13] that the Cartan matrix of  $B_0$  has no diagonal entry equal to 2. If  $G$  is one of  $\mathfrak{A}_7$ ,  $M_{11}$ ,  $M_{23}$ , or  $HS$ , then one checks from GAP [GAP13] that the Cartan matrix of  $B_0$  does not have the shape of Theorem 3.2(b) either.

In case (ii), then the Cartan matrix of  $B_0$  is computed in [Kun00, Table 2] and does not satisfy Theorem 3.2(b).

Next if  $G$  is one of the groups listed in Proposition 7.1(iii),(iv),(vii), or (ix), then it is proven in [KY10, Lemma 3.7] that  $B_0$  is Puig equivalent to  $B_0(kN)$ . But  $N$  has a non-trivial normal Sylow 3-subgroup, therefore all simple  $B_0(kN)$ -modules lie at the end of their components in  $\Gamma_s(B_0(kN))$  by Theorem 3.1(a), and therefore so do the simple  $B_0$ -modules via the latter Puig (Morita) equivalence.

In case (v), the decomposition numbers of  $B_0$  were computed by White and Okuyama-Waki. If  $q$  is even then we read from [Whi95, Table II] that each column of the decomposition matrix of  $B_0$  has at least 3 positive entries. If  $q$  is odd, then the decomposition matrix of  $B_0$  is given in [Whi90, Theorem 4.2] up to two parameters  $\alpha$  and  $\beta$ . But [OW98, Theorem 2.3] proves that  $\alpha \in \{1, 2\}$ . This is enough to see that each column of the decomposition matrix of  $B_0$  has at least 3 positive entries. Therefore in both cases all the diagonal entries of the Cartan matrix of  $B_0$  are at least 3.

In case (vi) and (viii), we proceed as follows. For  $n \in \{4, 5\}$  fixed, we may regard  $B_0(k\text{PSL}_n(q))$  as the principal block of  $\text{SL}_n(q)$  as  $3 \nmid |Z(\text{SL}_n(q))|$ . Then we check that the Cartan matrix of  $B_0(k\text{GL}_n(q))$  does not satisfy Theorem 3.2(b). To this end we use the information on the decomposition numbers of  $B_0(k\text{GL}_n(q))$  provided in [Jam90, Appendix I]. In both cases, it is enough to consider only the square submatrix  $\Delta_{n,0}$  of the decomposition matrix of  $B_0(k\text{GL}_n(q))$  whose rows are indexed by the unipotent characters. Both in case  $n = 4$  and  $n = 5$ , there are five modular characters in the principal block (using [FS82]) and

$$\Delta_{4,0} = \begin{bmatrix} (4) & 1 & & & \\ (31) & 1 & 1 & & \\ (2^2) & & 1 & 1 & \\ (21^2) & 1 & 1 & 1 & 1 \\ (1^4) & 1 & & 1 & 1 \end{bmatrix} \quad \Delta_{5,0} = \begin{bmatrix} (5) & 1 & & & \\ (32) & & 1 & & \\ (31^2) & 1 & 1 & 1 & \\ (2^21) & & 1 & 1 & 1 \\ (1^5) & 1 & & 1 & 1 \end{bmatrix}.$$

(See e.g. [KM00, Propositions 3.1 and 4.1].) It follows that the Cartan integers of  $B_0(\text{GL}_n(q))$  have lower bounds given by the entries of the following matrices:

$${}^T\Delta_{4,0}\Delta_{4,0} = \begin{bmatrix} 4 & 2 & 1 & 2 & 1 \\ 2 & 3 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad {}^T\Delta_{5,0}\Delta_{5,0} = \begin{bmatrix} 3 & 1 & 2 & 0 & 1 \\ 1 & 3 & 2 & 1 & 0 \\ 2 & 2 & 3 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Therefore the Cartan matrix of  $B_0(k\text{GL}_n(q))$  cannot satisfy Theorem 3.2(b), and we conclude that all simple  $B_0(k\text{GL}_n(q))$ -modules lie at the end of their AR-components.

Now, from the known values of the unipotent characters of  $GL_n(q)$ , we easily check using CHEVIE [Chv96] that the dimensions of the simple modules in  $B_0(kGL_n(q))$  are prime to 3. Hence they cannot be periodic by [Car79], as  $3^{(a-1)}$  must divide the dimension of any simple periodic module, where  $a =$  the  $p$ -rank of the group, but in our case  $a \geq 2$ . Therefore every simple  $B_0(kSL_n(q))$ -module lies at the end of its AR-component by [KMU00, Lemma 1.5].

Finally, if  $G = \mathrm{PSL}_2(3^n)$  for some integer  $n \geq 2$ , then the claim follows from Theorem 3.1(c) as  $G$  is a finite simple group of Lie type in defining characteristic.  $\square$

As a corollary we obtain Theorem E of the Introduction.

*Proof of Theorem E.* The claim now follows from Corollary D together with Proposition 7.2.  $\square$

**Acknowledgements.** Both authors gratefully acknowledge financial support and the hospitality provided by the Centre Interfacultaire Bernoulli (CIB) of the École Polytechnique Fédérale de Lausanne during the writing period of this article. The second author gratefully acknowledges financial support by the funding scheme TU Nachwuchsring Individual Funding 2016 granted to the second author by the TU Kaiserslautern. The first author is grateful to the hospitality of the Faculty of Mathematics of the TU Kaiserslautern. Finally, the authors would like to thank the referees for their careful reading and their many useful comments and pieces of advice.

#### REFERENCES

- [Alp76] J.L. Alperin, *Isomorphic blocks*, J. Algebra **43** (1976), 694–698.
- [ARS95] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras* (Cambridge Univ. Press, 1995).
- [Ben98] D.J. Benson, *Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras* (Cambridge Univ. Press, 1998).
- [BU01] C. Bessenrodt and K. Uno, *Character relations and simple modules in the Auslander-Reiten graph of the symmetric and alternating groups and their covering groups*, Algebr. Represent. Theory **4** (2001), 445–468.
- [Car79] J.F. Carlson, *The dimensions of periodic modules over modular group algebras*, Illinois J. Math. **23** (1979), 295–306.
- [Chv96] M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, *CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras*. *Appl. Algebra Engrg. Comm. Comput.* **7** (1996), 175–210.
- [Dad73] E.C. Dade, *Block extensions*, Illinois J. Math. **17** (1973), 198–272.
- [Dad77] E.C. Dade, *Remarks on isomorphic blocks*, J. Algebra **45** (1977), 254–258.
- [Due04] O. Düvel, *On Donovan’s conjecture*, J. Algebra **272** (2004), 1–26.
- [Erd90] K. Erdmann, *Blocks of Tame Representation Type and Related Algebras.*, Lec. Notes in Math. **1428** (Springer, 1990).
- [Erd95] K. Erdmann, *On Auslander-Reiten components for group algebras*, J. Pure Appl. Algebra **104** (1995), no. 2, 149–160.
- [FH93] P. Fong and M.E. Harris, *On perfect isometries and isotypies in finite groups*, Invent. Math. **114** (1993), 139–191.
- [FS82] P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups*, Invent. Math. **69** (1982), 109–153.
- [GAP13] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.6.5*, 2013.
- [GLS2] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups, Number 2* (Math. Surveys Monographs, Amer. Math. Soc., 1996).

- [GLS3] ———, *The Classification of the Finite Simple Groups, Number 3* (Math. Survey Monographs, Amer. Math. Soc., 1998).
- [HK99] A. Hida and S. Koshitani, *Morita equivalent blocks in non-normal subgroups and  $p$ -radical blocks in finite groups*, J. London Math. Soc. (2) **59** (1999), 541–556.
- [Hum06] J.E. Humphreys, *Modular representations of finite groups of Lie type*, London Math. Soc. Lec. Note Series **326** (Cambridge University Press, 2006).
- [Jam90] G. James, *The decomposition matrices of  $GL_n(q)$  for  $n \leq 10$* , Proc. London Math. Soc. (3) **60** (1990), 225–265.
- [Kaw97] S. Kawata, *On Auslander-Reiten components and simple modules for finite group algebras*, Osaka J. Math. **34** (1997), 681–688.
- [KMU00] S. Kawata, G. O. Michler and K. Uno, *On simple modules in the Auslander-Reiten components of finite groups*, Math. Z. **234** (2000), 375–398.
- [KMU01] ———, *On Auslander-Reiten components and simple modules for finite groups of Lie type*, Osaka J. Math. **38** (2001), 21–26.
- [Knö79] R. Knörr, *On the vertices of irreducible modules*, Ann. of Math. (2) **110** (1979), 487–499.
- [KM00] S. Koshitani and H. Miyachi, *The principal 3-blocks of four- and five-dimensional projective special linear groups in non-defining characteristic*, J. Algebra **226** (2000), 788–806.
- [KY10] S. Koshitani and Y. Yoshii, *Eigenvalues of Cartan matrices of principal 3-blocks of finite groups with abelian Sylow 3-subgroups*, J. Algebra **324** (2010), 1985–1993.
- [Kue90] B. Külshammer, *Morita equivalent blocks in Clifford theory of finite groups*, Astérisque **181–182** (1990), 181–182.
- [Kue95] B. Külshammer, *Donovan’s conjecture, crossed products and algebraic group actions*, Israel J. Math. **92** (1995), 295–306.
- [Kun00] N. Kunugi, *Morita equivalent 3-blocks of the 3-dimensional projective special linear groups*, Proc. London Math. Soc. (3) **80** (2000), 575–589.
- [Lan83] P. Landrock, *Finite Group Algebras and their Modules*, London Math. Soc. Lecture Note Series **84** (Cambridge University Press, 1983).
- [Las11] C. Lassueur, *Relative projectivity and relative endo-trivial modules*, J. Algebra. **337** (2011), 285–317.
- [NT88] H. Nagao and Y. Tsushima, *Representations of Finite Groups* (Academic Press, 1988).
- [OU94] T. Okuyama and K. Uno, *On the vertices of modules in the Auslander-Reiten quiver. II*, Math. Z. **217** (1994), no. 1, 121–141.
- [OW98] T. Okuyama and K. Waki, *Decomposition numbers of  $Sp(4, q)$* , J. Algebra **199** (1998), 544–555.
- [Vil59] O.E. Villamayor, *On the semisimplicity of group algebras II*, Proc. Amer. Math. Soc. **10** (1959), 21–27.
- [Whi90] D.L. White, *Decomposition numbers of  $Sp(4, q)$  for primes dividing  $q \pm 1$* , J. Algebra **132** (1990), 488–500.
- [Whi95] ———, *Decomposition numbers of  $Sp_4(2^a)$  in odd characteristics*, J. Algebra **177** (1995), 264–276.

SHIGEO KOSHITANI, CENTER FOR FRONTIER SCIENCE, CHIBA UNIVERSITY, 1-33 YAYOI-CHO, INAGE-KU, CHIBA, 263-8522, JAPAN.

*E-mail address:* `koshitan@math.s.chiba-u.ac.jp`

CAROLINE LASSUEUR, FB MATHEMATIK, TU KAISERSLAUTERN, POSTFACH 3049, 67653 KAISERSLAUTERN, GERMANY.

*E-mail address:* `lassueur@mathematik.uni-kl.de`